The Stefan Problem

by

L. I. Rubinsteĭn

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by

L. I. RUBENŠTEĬN

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Л. И. РУБИНШТЕЙН

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PREFACE

In recent years we have witnessed an intensive development of the Stefan problem, with which the author has long been concerned.

This monograph was originally intended to examine the important results concerned with the classical Stefan problem and its generalizations, up to the present time, but this program could not be carried out completely, first of all because the literature concerned with this problem is continuously increasing, and very important work, in which the Stefan problem is treated as a problem of numerical analysis, has been only partially published and partially announced in publications and at scientific conferences;[1] at the same time the work planned for our monograph was to a large extent complete. In this regard the numerical solution of problems of Stefan type is usually discussed in monographs in an extremely concise manner. We will limit ourselves to brief descriptions of finite difference algorithms which have been proposed for the solution of the Stefan problem (Part 2, Chapter VIII); as a rule we will not give a justification of the algorithm but refer the reader to the original literature. The author is, on the one hand, mainly interested in the physical formulation of a problem which reduces to some form of the Stefan problem (Part 1, Chapters I—IV) and, on the other hand, in the examination of general theoretical problems, i.e. problems of a qualitative character (Part 2, Chapters I—VII). In Supplement 3 we present illustrative examples of the numerical solution to problems examined in the first part. In the first two supplements we present certain known results which are used in the main parts of the book.

At the end of the book we include a bibliography of the problem. Unfortunately, many sources were not available to the author. It is probable that many works exist which have excaped his attention. Therefore we make no claim of completeness for the bibliography.

The author takes this opportunity to express his appreciation to his colleagues N. Avdoņins, M. Antimirov, A. Kuĭkis, H. Geĭman,

[1] We have in mind the joint work of B. M. Budak, F. P. Vasil'ev and A. B. Uspenskiĭ, as presented in [15] and announced in their articles in a number of reports of conferences on numerical analysis (Moscow State Univ., 1965).

E. Enikeeva, M. Zavorina, B. Martuzans, E. Pil̦enkova and A. Skroman, who checked the accuracy of the numerical solutions of the examples of Supplement 3 and aided the author in eliminating many errors, as well as to E. Riekstiņš and V. Abolinja, who read the manuscript of the monograph. The author expresses special thanks to the Rector of the Stučki Institute of the Latvian State University, Professor V. A. Šteĭnberg, and to the director of the University computing center, Docent È. I. Arin, without whose efficient aid the monograph could not have been brought to completion.

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INTRODUCTION

§1. Historical survey[1)]

The object of this study is to examine the process of diffusion of heat in a medium accompanied by a change of phase state or some process mathematically similar to it.

Apparently, the first work of interest in this area was the paper of G. Lamé and B. P. Clapeyron [88], published in 1831. In this work the problem was posed of determining the thickness of the solid crust generated by the cooling of a liquid filling the halfspace $x > 0$ under the influence of a constant temperature at the plane $x = 0$. The temperature of the liquid phase is initially everywhere equal to the crystallization temperature. Lamé and Clapeyron found that the thickness of the crust is proportional to the square root of the time but did not determine the coefficient of proportionality.[2)]

In 1889, J. Stefan in his work on the freezing of the ground [167] posed and solved the following two problems.[3)]

1. A material existing in two phase states (liquid and solid) and transmitting heat only by conduction fills the halfspace $x > 0$. At the initial time it is at the constant temperature $T_2 > 0$. At the surface $x = 0$ it is maintained at constant temperature $T_1 < 0$, under the effect of which there arises a crystallization, occuring isothermally for temperature $T = 0$ without supercooling, for which we neglect small volume effects.

2. The heat conducting material occupies the space $-\infty < x < \infty$. At the initial time the liquid phase fills the domain $0 < x < \infty$ at temperature $T_2 > 0$, while the solid occupies the domain $-\infty < x < 0$ at temperature $T_1 < 0$. The remaining conditions are the same as in the first problem.

In these cases it is required to determine the temperature $U_1(x, t)$

[1)] Below we do not confine ourselves to strict chronological order, but observe it only for results obtained by similar methods.

[2)] This fact was pointed out by M. Brillouin [13].

[3)] According to H. Weber [198], the solution to the first of these two problems was given by Franz Neumann in unpublished lectures delivered in Königsberg in the early sixties of the last century.

and $U_2(x,t)$ of the solid and liquid phases, and the position $x = y(t)$ of the boundary between the phases.

Calculation of the thermal balance yields, as shown by Stefan, the condition

$$\lambda\rho\frac{dy}{dt} = \left(k_1\frac{\partial}{\partial x}U_1 - k_2\frac{\partial}{\partial x}U_2\right)\Bigg|_{x=y(t)}. \tag{0.1.1}$$

Here λ is the latent heat of crystallization per unit mass; ρ is the density of the material in its original phase state, and k_1 and k_2 are the coefficients of conductivity corresponding to the solid and liquid phases, respectively.[4)]

Both problems examined by Stefan admit similarity solutions. In these cases $y = 2\alpha\sqrt{t}$, where α is determined by certain transcendental equations which we will not derive here.

In the same work of 1889, Stefan gave an analogous description of processes of neutralization for diffusion transport of material in a reaction zone [**168**], and evaporation and condensation [**170**]. Finally, in 1889 Stefan published his fourth work related to this problem [**169**]. In this work he examined the problem of melting of a layer of ice with initial temperature equal to zero, subject to the influence of the temperature $f(t)$ at the boundary $x = 0$.

This problem is solved in closed analytic form only for $f \equiv$ const. Stefan attempted to determine the thickness of the melted layer by the following method. The temperature $U(x,t)$ in this layer can be represented in the form of the series

$$U = f + \frac{x^2}{2a^2}f' + \frac{x^4}{4!a^4}f'' + \cdots + xF + \frac{x^3}{3!a^2}F' + \cdots \tag{0.1.2}$$

(a^2 is the coefficient of thermal conductivity of the material in this layer).

The unknown function F is determined from the condition that $U \equiv 0$ at the interphase boundary $x = y(t)$, which gives

$$0 = f + \frac{y^2}{2a^2}f' + \frac{y^4}{4!a^4}f'' + \cdots + yF + \frac{y^3}{3!a^2}F' + \cdots, \tag{0.1.3}$$

[4)] Below, condition (0.1.1) and its generalization to spatial problems will be called the Stefan condition.

plus condition (0.1.1), which in the present case is of the form

$$\frac{\lambda\rho}{k}y' = -\frac{\partial}{\partial x}U\Big|_{x=y} = -\left(\frac{y}{a^2}f' + \frac{y^3}{3!a^4}f''' + \cdots + F + \frac{y^2}{2a^2}F' + \cdots\right). \tag{0.1.4}$$

In order to obtain the first approximation, Stefan eliminated F from (0.1.3) and (0.1.4) so as to give

$$\frac{\lambda\rho}{k}y\cdot y' = f - \frac{y^2}{2a^2}f' - \cdots - \frac{y^3}{3a^2}F' - \cdots. \tag{0.1.5}$$

Ignoring every term in the right side of (0.1.5) except the first, Stefan obtained

$$\frac{\lambda\rho}{k}yy' = f. \tag{0.1.6}$$

In order to obtain the second approximation, Stefan used the obvious equation

$$\left(\frac{\partial}{\partial t} + y'\frac{\partial}{\partial x}\right)U|_{x=y(t)} = 0,$$

which gives

$$\frac{\lambda\rho}{k}y'^2 = f' + \frac{y^2f'''}{2a^2} + \cdots + yF' + \frac{y^3F'''}{3!a^4} + \cdots. \tag{0.1.7}$$

Eliminating F' from (0.1.5) and (0.1.7) and neglecting all terms containing powers of f and F of order greater than the first, Stefan obtained

$$\frac{\lambda\rho}{k}yy'\left(1 + \frac{yy'}{3a^2}\right) = f - \frac{y^2}{6a^2}f'. \tag{0.1.8}$$

Obviously it is impossible to construct an explicit solution to the problem by a method of this type. Nevertheless the first approximation (0.1.6) gives a very good approximation to the solution. Namely, assuming that $f' \geqq 0$ in the vicinity of $t = 0$, it can be shown by the maximum principle that $\lambda\rho yy' \leqq kf$, and moreover,

$$\lim_{t\to 0}\frac{\lambda\rho yy'}{kf} = 1. \tag{0.1.9}$$

From 1891 to 1931 there was no serious publication devoted to the problem.[5)]

In 1931 L. S. Leĭbenzon, in a course on mechanics in the oil industry [92], proposed a method of approximation for the solution of the Stefan problem, which is very effective in many cases. It consisted of replacing the true temperature distribution within each phase by a quasi-stationary solution, i.e. one which satisfies the Laplace equation with respect to the space coordinates in a domain with a boundary varying in correspondence with the solution of Stefan.

In 1939 Leĭbenzon applied this method to different variants of the calculation of the time of solidification of the spherical earth from its original molten condition [93]. Still earlier, this method was successfully applied by S. S. Kovner for the solution of the problem of a thawed patch on a half-plane [81] and to the freezing of a sphere [82]. In 1947 Leĭbenzon's method was used by A. N. Tihonov and E. G. Švidkovskiĭ for the solution of a metallurgical problem on the formation of a continuous ingot [180]. Finally, we find in the work of I. A. Carnyĭ [17] a proposal for refinement of Leĭbenzon's method, making it possible to take partial account of the influence of heat capacity of both phases, which is not possible in the original method.

The first attempt to approach the precise problem of Stefan was made by Brillouin in a lecture given in 1929 at the Institut Henri Poincaré, and published in 1931 [13]. Brillouin proposed the following method for the reduction of the problem to a system of nonlinear integro-differential equations. Let ϕ_1 and ϕ_2 be the temperatures of the solid and liquid phases at the initial moment, and let

$$E(x,t) = (2\sqrt{\pi t})^{-1}\exp(-x^2/4t)$$

be the fundamental solution to the heat equation. Brillouin sought the temperature u_i of phase i in the form

$$u_i(x,t) = \int_{-\infty}^{\infty} \Phi_i(\xi)\, E(x-\xi, a_i^2 t)\, d\xi \qquad (i = 1, 2),$$

where Φ_i coincides with ϕ_i in the domain of definition of the latter.

[5)] The work of Stefan [169] mentioned above attracted the attention of geophysicists and caused the appearance of several unsuccessful attempts to solve the more general problems. A bibliography of this specifically geophysical literature is given by B. P. Veĭnberg [189].

Subjecting u_i to boundary and initial conditions, on the one hand, and to the Stefan conditions (0.1.1) on the other, we obtain a system of singular integro-differential equations of the first kind for the extension of the ϕ_i beyond the boundary of their original domain of definition and for the boundary $x = y(t)$ between the phases.

It is not necessary for us to prove here that this method is not feasible, at least not without extremely strong restrictions on the boundary data and the initial data of the problem. Brillouin himself wrote out the integral equations but avoided trying to look for a method of solving them, assuming that the difficulties were insurmountable. Thus Brillouin reported on his method in order to raise interest in the Stefan problem, rather than to solve it.

The next step towards the solution of the problem was made by A. Huber [61]. Huber's method, which is a generalization of the method of polygonal approximation of Cauchy-Lipschitz for the one-dimensional Stefan problem, consists of the following. The interval of time $[0, T]$, on which a solution is sought to the problem, is divided by means of points $t_0 = 0 < t_1 < \cdots < t_n = T$ into subintervals $\Delta_k = (t_k, t_{k+1})$. The velocity $\dot{y}(t)$ of the advancing boundary separating the phases in Δ_k is assumed constant and equal to the velocity determined by the Stefan condition for the moment $t = t_k$ and by the temperatures of the phases at that moment.

The temperature is determined by the boundary conditions and is found for $t = t_k$ in the following way. The domain of existence of each phase in the interval Δ_k is taken into the segment $[0,1]$ by an affine transformation. Here Huber made use of the invariance of the heat equation under an Appell transformation [53] and of the representation of the solution of the first boundary problem on the segment by means of the Green's function.

But this procedure, as Huber noted, is very cumbersome. Presumably it was for this reason that the author did not even try to give it a strict justification. We note, however, that the method was unnecessarily complicated by Huber's being unaware, it seems, of the expression for the Green's function of the first boundary problem for the one-dimensional equation of heat conduction in a domain with rectilinear boundary. An expression of this Green's function, very convenient for the applications, was constructed by P. V. Solov'ev in the same year 1939 [165]. With this Green's function, it is possible to write out the solution on each

interval of the subdivision, without having to use the Huber transformation.[6)]

The proof of existence and uniqueness of the solution to the one-dimensional Stefan problem in the case of more general curves and initial conditions was realized for the first time in 1947 in our dissertation [124]. The method which we used reduces to the transformation of the domain of existence of each phase to the interval $[0,1]$, by means of a reduction of the problem to a system of nonlinear integral equations of mixed type (Fredholm type with respect to the spatial coordinate and Volterra type with respect to time). We obtain a system solvable by Picard's method of successive approximations, where moreover the convergence process is guaranteed in the small, i.e. in some neighborhood of $t = 0$. The diameter of this neighborhood is estimated in terms of the boundary and initial temperature conditions and their derivatives, and also, as is particularly important, by the diameters of the domains of initial existence of each phase, supposing them different from zero. If these diameters tend to zero the time interval on which the convergence of the iterative process is proved also converges to zero.

This method of solution of the Stefan problem yields an algorithm which has little effectiveness, since an iterative procedure requires the calculation of double integrals not fundamentally connected with the problem. Therefore in the same year (1947) we suggested two methods of reducing the Stefan problem to integral equations of Volterra type [125], based on the direct use of the heat potential. Here also the proof of existence and uniqueness of the solution is given in the small. However, the algorithm of this method is considerably more advantageous, since it does not demand the calculation of multiple integrals. Its essential feature consists in the separation of the problem of determining the position of the interphase boundary from the problem of calculating the temperature distribution within each phase. The integral equations are formed for the variables entering in the Stefan conditions, and only for them. The calculation of the temperature distribution within each phase reduces, after the solution of these equations, to simple quadratures.

[6)] A method similar to the method of Huber was proposed by A. Dacev [22] in 1947 and consists of the approximation of the unknown boundary by a broken line whose components are parallel to the coordinate axes. Unfortunately, every proof given by Dacev is invalid [40], [133], a stricture which is especially relevant to the subsequent publications of Dacev [26], [27] on the planar and spatial Stefan problems. Thus the method of Dacev must be considered as an unsupported heuristic suggestion. Numerical testing of this method has apparently not been carried out by anyone.

In 1948-51 this method was used for the investigation of the steady state solution [126], the first investigation of the velocity of advance of the interphase boundary in the case of a 2-phase Stefan problem on an interval with one-phase initial condition [127], the solution of the problem with cylindrical symmetry [128] and the investigation of the diffusion of heat in a multi-phase multi-layered medium with non-isothermal phase transitions in each of the layers [129].

In a 1948 note [130] we outlined as well a proof of the existence of a solution of the two-phase Stefan problem on a segment initially containing one phase, based on the proof of the possibility of constructing a solution to the 2-phase initial value problem in the large, i.e. on an arbitrary preassigned time interval,[7] and in 1951 [132] we studied the asymptotic behavior of the solution to the Stefan problem for $t \to \infty$.

Investigations similar to those of [124]—[132] were carried out by I. I. Kolodner [80] and A. Friedman [40]—[43]. In [40] Friedman gave an extended proof of the theorems briefly stated by us in [124], and in [41] and [42] he carried out a detailed investigation of the problems of evaporation of a spherical drop and dissolution of a gas bubble in a liquid. Finally, in [43] Friedman examined the single-phase Stefan problem with a degenerate initial domain consisting of a point and monotonically varying boundaries.

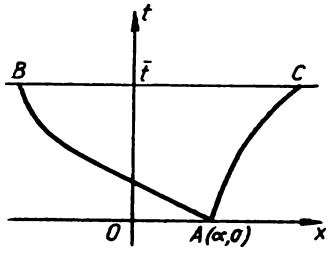

FIGURE 1

In [80] Kolodner examined the Stefan problem in the following form. Let there be given a domain D_R in the (x,t) plane, bounded by a free curve $AB: x = R(t)$, a given curve $AC: x = R_0(t)$ and the characteristic line $t = E$ (Figure 1). It is required to determine $U(x,t)$ and $R(t)$, satisfying the conditions

$$\begin{aligned} &U_{xx} = U_t \text{ in } D_R; \qquad U = f(t) \text{ on } AB; \\ &\Gamma U = 0 \text{ on } AC; \qquad U_x = g(t) \text{ on } AB. \end{aligned} \tag{0.1.10}$$

Here Γ is an operator which should be given so that the boundary problem

$$w_{xx} - w_t = 0; \quad \Gamma(w + v) = 0$$

is uniquely solvable in the domain D, bounded from the right by the

[7] The proof offered in [130] contained technical errors and was not rigorous; it was essentially redone in [131].

curve AC and extending to infinity, for arbitrary differentiable functions v given on AC.

Using the heat potential and the method of extension of the solution across the boundary of the domain, Kolodner proved a number of reduction theorems, leading finally to the integral equations constructed in [125]. In addition, in [80] he formulated a number of interesting problems of type (0.1.10).

We note, finally, that in 1958 we carried out an examination of the method of [125] as a method for the numerical solution of a one-dimensional Stefan problem [135], and in 1963 this method was applied for the solution in the small of the one-dimensional Stefan problem for the heat-conduction equation with a continuously distributed heat source, nonlinearly dependent on the temperature, on its derivative with respect to the space coordinate, and on the interphase boundary and the velocity of its advance, for nonlinear conditions both on the given and on free boundaries [136].

In 1951 there was published an important work of G. W. Evans [34], containing a theorem of existence of the solution to the one-phase single-layer Stefan problem in the following form:

$$\begin{gathered} U_{xx} = U_t; \quad 0 < x < x(t); \quad U|_{x=x(t)} = 0; \\ U_x|_{x=0} = -1; \quad \dot{x}(t) = -U_x|_{x=x(t)}; \quad x(0) = 0. \end{gathered} \tag{0.1.11}$$

Integrating the equation $U_{xx} = U_t$ on the domain $0 < x < x(t)$ and using the boundary condition of the problem, Evans obtained

$$x(t) = t - \int_0^{x(t)} U(x,t)\,dx. \tag{0.1.12}$$

Since for the given boundary $x = x(t)$ the solution to the boundary problem (0.1.11) is determined uniquely, the operator equation

$$Rx = t - \int_0^{x(t)} U(x,t)\,dx \tag{0.1.12*}$$

determines a transformation of the set of all smooth functions $x(t)$, $x(0) = 0$, into itself. Let $U(x,t,h)$ be the solution to the boundary problem under consideration for given boundary $x = h(t)$. Evans proved, using monotonicity of $h(t)$, the convergence of the iterative process

$$x_n(t) = t - \int_0^{x_{n-1}(t)} U(x,t;x_{n-1})\,dx \tag{0.1.13}$$

to the solution of the original problem.

The problem considered by Evans can be solved by a method proposed in [125]—[131], with evident simplification, and from this point of view it appeared that the Evans method does not give anything new. However, further investigation showed its greater intrinsic merit, consisting, first of all, in that it can be extended to the equation of heat conduction with quasilinear principal part. An example of the use of Evans' method was given in a later work of W. T. Kyner [85], in which he proved the existence and uniqueness of the solution to the single-phase Stefan problem, posed for a general quasilinear equation of parabolic type. Unfortunately, the greatly increased generality of Kyner's result is achieved at the price of the loss of an effective method. The existence theorem proved by Kyner is based on the use of the Schauder fixed-point principle and therefore is essentially impractical.[8] Kyner's work is relevant to the investigations of L. I. Kamynin [74], relating to filtration problems, on which we remark below.

The work of Evans was followed by the investigations of G. Sestini [157]—[163], using Evans' method for the solution of more general problems, and in particular the two-phase problems of J. Douglas and T. M. Gallie [32]. The first of these contained the proof of a uniqueness theorem, lacking in the work of Evans and Sestini;[9] and the second developed a finite-difference method for the solution of a single-phase Stefan problem in the form analyzed by Evans. This method consists essentially of utilizing an equation of type (0.1.12). Douglas and Gallie solved the problem of Evans by constructing a difference algorithm such that at each time step the unknown boundary $x = x(t)$ moves one mesh node. To this end they constructed an iteration process, determining the time mesh by using an equation of the type (0.1.12). For the proof of their method Douglas and Gallie, like Evans and Sestini, used the fact that the boundary condition considered by them guarantees a monotonic boundary curve. Later, in the work of B. M. Budak, F. P.

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[8] An account of Kyner's results is given below in Part 2, Chapter VII, §1.

[9] The proof of the uniqueness theorem presented in the work of Evans is based on the examination of two systems of solutions to problem (0.1.11) with boundary $x = y(t)$ and $x = z(t)$ respectively, such that the difference $z(t) - y(t)$ is monotonic in a neighborhood of $t = 0$. Thus uniqueness of the solution is proved only under the assumption of an analytic boundary, and the existence of a solution only in the class of domains with smooth boundary. Analogous results are given in the work of Sestini. Here Evans noted the insufficiency of the proof, while on the other hand, Sestini assumed it completely correct. In reply to the critical remark of J. Douglas (in [31]) Sestini published two notes, containing invalid attempts to correct his original proof of the uniqueness theorem [158], [159].

Vasil'ev and A. B. Uspenskiĭ [187], [188], [15] the method of Douglas and Gallie was generalized to the case of a single-phase Stefan problem for a general quasilinear equation.

A second finite-difference method for the solution of the Stefan problem for the equation of heat conduction was proposed in 1960 by M. Lotkin [96].[10] His method was based on transforming the domain of existence of each phase into an interval. In his work Lotkin examined a single-phase problem of the type examined by Evans. Later, Lotkin's method was extended by Budak, Vasil'ev and Uspenskiĭ to the case of a general quasilinear parabolic equation and even to the case of the Stefan problem in space [15]. In the latter case an algorithm was constructed for the solution of the problem, assuming its existence and sufficient smoothness. We note that the existence of such a solution remains up till now unproven even for the case of the Fourier equation of heat conduction with constant coefficients.

In 1958 V. G. Melamed developed an efficient method for reducing the two-phase Stefan problem on an interval to a denumerable system of differential equations [99]. This method consists of the following. Let $u(x,t)$, $y(t)$ be the solution of the problem

$$\begin{gathered} u_{xx} = u_t,\ 0 < x < y(t),\ t > 0; \quad u|_{x=0} = f(t); \\ u|_{x=y(t)} = 0; \quad u|_{t=0} = \phi(x); \quad \dot{y}(t) = u_x|_{x=y(t)}; \\ y(0) = l > 0; \quad \phi(l) = 0; \quad \phi(0) = f(0), \end{gathered} \tag{0.1.14}$$

where ϕ and f are of constant sign. Further, let

$$v(x,t) = u(x,t) - f(t)\frac{y(t) - x}{y(t)}. \tag{0.1.15}$$

Then v can be expanded in a Fourier sine series

$$v(x,t) = \sum_{n=1}^{\infty} A_n(t) \sin\frac{n\pi x}{y(t)}. \tag{0.1.16}$$

Subjecting v and y to the boundary and initial conditions of the problem, Melamed constructed a countable system of ordinary differential equations for the determination of y and A_n $(n = 1, 2, \cdots)$. He proved that the truncation of the system to one containing the first n coefficients A gives an approximate solution of the problem which for $n \to \infty$ converges to the solution of the original problem on a sufficiently small time interval, the length of which tends to zero with the length l of the

[10] A justification of the method in the work of M. Lotkin was not given.

original interval of existence of the phase under consideration. In subsequent articles of Melamed [100] and I. V. Frjazinov [47] this method was extended to other problems of the same type. The essential merit of Melamed's method is the ease with which the computational process can be programmed for the ÈVCM computer. The method of Melamed is one of the practical working methods for the numerical solution of problems of Stefan type.

In all the above publications the Stefan problem is treated as a problem of determining the unknown boundary. However, A. N. Tihonov and A. A. Samarskiĭ have indicated the feasibility of a basically different approach [179]. Namely, they note that in the limit for $\epsilon \to 0$ the first of the problems considered by Stefan (see above, pp. 1 and 2) results in the boundary problem

$$\frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right) = \rho c(u)\frac{\partial u}{\partial t}; \quad 0 < x < \infty; \tag{0.1.17}$$

$$u|_{x=0} = u_1 = \text{const} < 0; \quad u|_{t=0} = u_2 = \text{const} > 0,$$

if in this problem we choose

$$k(u) = \begin{cases} k_1 = \text{const} > 0 \text{ for } u < 0; \\ k_2 = \text{const} > 0 \text{ for } u > 0; \end{cases} \quad \rho = \text{const} > 0,$$

$$c(u) = \begin{cases} c_1 = \text{const} > 0 \text{ for } u < -\epsilon; \\ c_2 = \text{const} > 0 \text{ for } u > +\epsilon; \end{cases} \quad \int_{-\epsilon}^{\epsilon} c(u)\,du = \lambda. \tag{0.1.18}$$

A similar treatment is carried over in a natural way to the problem with a nonsimilarity formulation. For this purpose it is necessary to introduce the functions

$$a(u) = \int_{u_0}^{u} c(z)\,dz + b(u); \quad b(u) = \begin{cases} 0; & u < 0; \\ \lambda; & u > 0, \end{cases} \tag{0.1.19}$$

reducing the heat conduction equation to a quasilinear equation of the form

$$\frac{\partial a(u)}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{0.1.20}$$

and considering the usual boundary value problem for it.

In apparent independence of the work of Tihonov and Samarskiĭ, such a treatment of the Stefan problem was undertaken by E. L. Albasiny [2], who in 1956 suggested an algorithm for the numerical solution of the Stefan problem based on reduction to an equation of the form (0.1.20). Later, on the basis of this method, S. L. Kamenomostskaja

[73] and O. A. Oleĭnik [107] introduced the notion of a generalized solution of the Stefan problem, whose uniqueness and existence was guaranteed in the class of measurable bounded functions. They proved that the generalized solution coincides with the classical one if the latter exists.

Kamenomostskaja considered the classical quasilinear heat conduction equation. She constructed the generalized solution by the use of an explicit difference scheme. Oleĭnik generalized the result of Kamenomostskaja to the case of the Stefan problem for general quasilinear equations of parabolic type. For this purpose she made use of an explicit difference scheme, with a method of smoothing of the coefficients developed earlier [108]. This permits the use of an economical implicit difference scheme, making Oleĭnik's method suitable for the numerical solution of the Stefan problem, particularly in the spatial case, when the other methods prove to be untenable.

Oleĭnik considered the question of the convergence of her method, but not the question of the actual numerical solution of the problem. From this point of view her method was examined in the previously cited paper of Budak, Vasil'ev and Uspenskiĭ [15]. In particular, this work contains useful suggestions for a method of smoothing the coefficients which yields a rather accurate numerical result (see also [14], [153]).

The method of Kamenomostskaja and Oleĭnik has very great generality. It was developed for general quasilinear equations of parabolic type, and also is valid not merely in one dimension, but in the spatial case. For the analysis of the problem by this method it makes no difference how many phases are involved in the process nor whether they are preserved for the whole time. Such great generality is naturally attained at the cost of a considerable amount of valuable information. It is a fact of essential importance that the method of Oleĭnik and Kamenomostskaja yields absolutely no information about the structure of the interphase boundary nor about the sense in which a generalized solution of the problem satisfies the Stefan conditions [109]. Not less essential is the fact that the application of the method of Kamenomostskaja-Oleĭnik is limited, apparently, to the case of isothermal phase transitions. In the same way this method is inapplicable to the solution of a one-phase Stefan problem when the Stefan condition (0.1.1) is replaced by a condition of the form

$$\frac{dy}{dt} = f(t) + k\frac{\partial u}{\partial x}\bigg|_{x=y(t)}, \tag{0.1.21}$$

a condition which is frequently encountered, particularly in applications to processes of phase transition with instantaneous removal of the generated phase. Such, for example, are problems of ablation from a rigid body in a gas stream flowing at high velocity.

So far we have reviewed work relating to problems of the construction of precise solutions of various versions of the Stefan problem. We will now mention two interesting articles whose purpose is to construct approximate solutions of the problem. The first is that of Bruno Boley [11], using an idea of upper and lower functions for constructing approximations to the solution of the problem; the second is that of I. Tadjbakhsh and W. Liniger [173], which examines one variant of a one-dimensional Stefan problem, introducing in the Stefan conditions a small parameter and expanding the solution in the form of a series with respect to this parameter. Tadjbakhsh and Liniger did not try to prove the convergence of their method, but showed that the two first approximations give qualitatively good and quantitatively satisfactory approximations to the exact solution.

Up to now we have discussed problems arising essentially from the classical thermal problem examined by Stefan [167]. In our treatment of such problems it was essential that we neglect changes in density under the phase transition. This made it possible to eliminate the necessity of examining hydrodynamical questions, since a change in density for a phase transition inevitably causes the onset of convection currents in the liquid phase even when the liquid phase can be assumed incompressible and its thermal expansion can be neglected. This was noted by P. L. Chambré in 1956 [18] in an examination of crystallization due to diffusion in a supercooled medium under the assumption that the generated solid phase remains isothermal at all times during the process. Chambré makes this assumption in order to be able to fix his attention solely on the liquid phase and to construct a similarity solution in the one-dimensional case as well as in the case of spherical and cylindrical symmetry. Here Chambré assumed that in the liquid phase there is no pressure gradient.[11)]

The numerous works in which the Stefan problem is treated as a hydrodynamical problem are linked to problems of ablation of a rigid body in a gas moving at high velocity. The majority of these works, in which the problem is treated as one-dimensional, do not contain

[11)] See Part 1, Chapter IV, §1.

original mathematical material. However, one finds many applications of significance. As it is not possible to halt our discussion in order to examine this work in detail, we merely indicate some references: [56], [149], [151], [172], [181]—[184].

A set of problems essentially different from the classical Stefan problem, or the problem of the unknown boundary for an equation of parabolic type, arises in the theory of filtration. The first such problem was formulated by N. N. Verigin [191] in 1952. The problem in [191] concerned the forcing of an almost incompressible liquid into the ground through a tunnel. If $x = y(t)$ denotes the boundary of penetration of the liquid in the ground due to the forcing, p denotes the pressure and k the coefficient of filtration, then one may impose conditions of continuity of p and $k\,\partial p/\partial x$ for passage across the boundary. On the other hand, the extent of penetration of the liquid in unit time clearly satisfies the equation

$$\alpha \frac{dy}{dt} = -k \frac{\partial p}{\partial x}\bigg|_{x=y(t)}, \tag{0.1.22}$$

where α is proportional to the porosity of the ground. Finally, in the domain filled with liquid and initially saturated, the pressure is subject to the equation of piezoconductivity, formally identical to the equation of heat conduction. Thus the problem examined by Verigin is found to be analogous to the usual double-layer heat problem, but with a conjugacy condition on the unknown boundary.

The problem examined in [191] admits a similarity solution. In a formulation not admitting a similarity solution, the problem of Verigin was examined by us in 1956 [138]. Later the same problem was examined by L. I. Kamynin [74]—[76], who gave an analysis of it for the case of dependence of the coefficients of piezoconductivity on the spatial coordinates and on time, in the domain in each of the filtration phases.

We will later examine the problem of determining the solution of the Fourier equation in a domain with unknown boundary, when the values of the unknown function and its normal derivative are given on this boundary. This problem, also arising in connection with some problems in the theory of filtration, was examined in one dimension by T. D. Ventcel′, using direct methods, in [190]. Problems of this type are essentially different from those examined above and our analysis is not applicable to them.

Problems of the Stefan type naturally arise in the study of viscoplastic flows as well. Several such problems were examined by A. I. Safrončik [147], [148], using the reduction theorems of Kolodner [80].

In concluding this survey we mention the formulation of the problem

inverse to the problems of Stefan and Verigin, in the work of G. A. Martynov [97], [98] and ourselves [139]. The general theory of such problems has not yet been developed, but is probably not far in the future. The inverse problem of Stefan type belongs to the category of incorrect problems, which have undergone intensive development in recent years, and therefore one can expect that it will soon be subjected to a careful examination.

§2. Notation and terminology

We will make use of the following definitions and notation.

A_1. A parabolic domain D in the plane (x,t) is a domain of the form

$$D = \{X_1(\tau) < x < X_2(\tau); \quad t_0 < \tau \leqq t \leqq T\} \tag{0.2.1}$$

under the condition that the interval $X_1(t) < x < X_2(t)$ refers to the set of interior points of D for any $t > t_0$ under consideration. Thus the boundary of a parabolic domain D consists of segments of the arcs $x = X_i(t)$ $(i = 1,2)$, and a segment of the characteristic $t = t_0$ lying between the points $(X_i(t_0), t_0)$ $(i = 1,2)$. Here $t > t_0$ is an arbitrary value of the time under consideration, with $t \leqq T$. When it is necessary to emphasize the type of boundary of D, we will write $D_{t,T}$ or $D(X_1, X_2, t_0, T)$ in place of D.

A_2. Let a be a positive constant. We will write

$$L_a = a^2 \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}. \tag{0.2.2}$$

Let $u(x,t)$ be a function defined in a parabolic domain D; suppose that it is continuous in the closure of D, uniformly bounded in D, and is such that u_t and u_{xx} are continuous throughout D except possibly at the points $(X_i(t_0), t_0)$ $(i = 1,2)$. We then say that $u(x,t)$ is superparabolic, parabolic, or subparabolic according as

$$L_a u \leqq 0; \quad L_a u = 0 \quad \text{or} \quad L_a u \geqq 0 \tag{0.2.3}$$

at all points of D. Where no misunderstanding can arise, we simply use the expressions superparabolic, parabolic or subparabolic function.[12)]

Instead of L_a we will generally write L_1 for $a^2 = 1$ and L_2 for $a^2 \neq 1$.

[12)] If $a = 1$, the subscript a in the cases of superparabolicity, etc. will be omitted. For a more general definition of the concepts of superparabolicity and subparabolicity, see I. G. Petrovskiĭ [112], and also [113]. The term "parabolic domain" was introduced by B. Pini [113].

A_3. The fundamental solution of the equation $L_1 u = 0$ will be denoted by $E(x,t)$ and is given by

$$E(x,t) = (2\sqrt{\pi t})^{-1}\exp(-x^2/4t). \tag{0.2.4}$$

The Green's functions of the first, second and third problems of the equation $L_1 u = 0$ on the halfline $x > 0$ are denoted by g_1, g_2 and g_3 respectively. These are given by

$$g_1(x,\xi,t) = E(x-\xi,t) - E(x+\xi,t), \tag{0.2.5}$$

$$g_2(x,\xi,t) = E(x-\xi,t) + E(x+\xi,t), \tag{0.2.6}$$

$$g_3(x,\xi,t) = g_2(x,\xi,t) - 2h\int_0^\infty \exp(-h\zeta)E(x+\xi+\zeta,t)\,d\zeta. \tag{0.2.7}$$

The Green's functions of the first and second boundary problems for the equation $L_2 u = 0$ on the halfline $x < 1$ are denoted by G_1 and G_2 respectively. Thus

$$G_1(x,\xi,t) = E(x-\xi,a^2t) - E(x+\xi-2,a^2t), \tag{0.2.8}$$

$$G_2(x,\xi,t) = E(x-\xi,a^2t) + E(x+\xi-2,a^2t). \tag{0.2.9}$$

The notation for Green's function for other domains or equations will be introduced in the text.

A_4. Along with the Green's function g_3 we will use the function

$$g_4(x,\xi,t) = g_1(x,\xi,t) + 2h\int_0^\infty \exp(-h\zeta)E(x+\zeta+\xi,t)\,d\zeta, \tag{0.2.10}$$

related to g_3 by the equations

$$\frac{\partial g_3}{\partial x} = -\frac{\partial g_4}{\partial \xi}; \quad \frac{\partial g_3}{\partial \xi} = -\frac{\partial g_4}{\partial x}. \tag{0.2.11}$$

A_5. Any two Green's functions g and g^* related by equations (0.2.11) are called conjugate. Thus g_1 and g_2, g_3 and g_4, G_1 and G_2 are conjugate functions.

Let g and g^* be conjugate Green's functions for an equation $L_a u = 0$, defined in some domain D. Furthermore, let $x(t)$ and $\xi(\tau)$ be two arbitrary differentiable functions and let $g(x(t),\xi(\tau),t-\tau)$ and $g^*(x(t),\xi(\tau),t-\tau)$ be defined in D. Then in D we have the equations

$$
\begin{aligned}
a^2\frac{\partial^2}{\partial x\partial\xi}g &= -\left(\frac{d}{dt}-\frac{dx}{dt}\frac{\partial}{\partial x}\right)g^* = \left(\frac{d}{d\tau}-\frac{d\xi}{d\tau}\frac{\partial}{\partial\xi}\right)g^*;\\
a^2\frac{\partial^2}{\partial x\partial\xi}g^* &= -\left(\frac{d}{dt}-\frac{dx}{dt}\frac{\partial}{\partial x}\right)g = \left(\frac{d}{d\tau}-\frac{d\xi}{d\tau}\frac{\partial}{\partial\xi}\right)g.
\end{aligned}
\tag{0.2.12}
$$

When referring to equations (0.2.11) and (0.2.12) we will assign appropriate subscripts to the conjugates g and g^*.

A_6. In the text, a dot over the symbol for a function of one variable denotes the derivative, with d/dt for the total derivative and $\partial/\partial t$ for the partial derivative. Thus for any differentiable functions y, f and Φ

$$
\begin{aligned}
\dot{y}(t) &= \frac{dy}{dt};\quad \frac{\partial}{\partial t}\Phi(x,t,f(x,t))\,|_{x=x(t)} = \left(\frac{\partial}{\partial t}+\frac{\partial f}{\partial t}\frac{\partial}{\partial f}\right)\Phi;\\
\frac{d}{dt}\Phi(x,t,f(x,t))\,|_{x=x(t)} &= \frac{dx}{dt}\left(\frac{\partial}{\partial x}+\frac{\partial f}{\partial x}\frac{\partial}{\partial f}\right)\Phi+\left(\frac{\partial}{\partial t}+\frac{\partial f}{\partial t}\frac{\partial}{\partial f}\right)\Phi.
\end{aligned}
\tag{0.2.13}
$$

A_7. The following notation is used in Part 2, Chapter I:

$$
\begin{aligned}
&I_{1,i}(x,t,\xi,\tau,a\,|\,f(x,t,\xi,\tau))\\
&\quad = a^2\int_0^t f(x,t,\xi,\tau)\,E(x+(-1)^i\xi,a^2(t-\tau))\,d\tau,
\end{aligned}
\tag{0.2.14$_1$}
$$

$$
\begin{aligned}
&I_{2,i}(x,t,\xi,\tau,a\,|\,f(x,t,\xi,\tau))\\
&\quad = a^2\int_0^t f(x,t,\xi,\tau)\frac{\partial}{\partial\xi}E(x+(-1)^i\xi,a^2(t-\tau))\,d\tau,
\end{aligned}
\tag{0.2.14$_2$}
$$

$$
I_{3,i}(x,t,\xi,\alpha,\beta,a\,|\,f(x,t,\xi)) = \int_\alpha^\beta f(x,t,\xi)\,E(x+(-1)^i\xi,a^2t)\,d\xi.
\tag{0.2.14$_3$}
$$

$$
\begin{aligned}
&I_{4,i}(x,t,\xi,\tau,\alpha,\beta,a\,|\,f(x,t,\xi,\tau))\\
&\quad = \int_0^t d\tau\int_\alpha^\beta f(x,t,\xi,\tau)\,E(x+(-1)^i\xi,a^2(t-\tau))\,d\xi,
\end{aligned}
\tag{0.2.14$_4$}
$$

$$
\begin{aligned}
&I_{5,i}(x,t,\xi,\tau,\alpha,\beta,a\,|\,f(x,t,\xi,\tau))\\
&\quad = \int_0^t d\tau\int_\alpha^\beta f(x,t,\xi,\tau)\frac{\partial}{\partial\xi}E(x+(-1)^i\xi,a^2(t-\tau))\,d\xi.
\end{aligned}
\tag{0.2.14$_5$}
$$

Part One

PHYSICAL PROBLEMS REDUCING TO PROBLEMS OF THE STEFAN TYPE

CHAPTER I

DIFFUSION OF HEAT BY CONDUCTION IN A MEDIUM WITH A CHANGE OF PHASE STATE

§1. The Stefan conditions

We now examine a medium transmitting heat by conduction and existing in two phase states characterized by their temperatures θ_i, coefficients of heat conduction k_i, specific heats c_i and densities ρ_i $(i = 1, 2)$. It will be assumed that the transition from phase 1 to phase 2 is accompanied by an absorption of heat. The change of density under the phase transition will be neglected.

Let $D_{i,t}$ be the domain containing the material in the ith phase at time t, and let S_t be the boundary between $D_{1,t}$ and $D_{2,t}$. The surface S_t is assumed to be smooth. S_t is a phase transition surface, across which discontinuities occur in the thermal characteristics k_i and c_i of the medium. At the same time it can be considered as a surface serving as a heat source for the medium. In fact, let M be a point of S_t with $\bar{n}_M$ the exterior normal to S_t at M relative to $D_{1,t}$. We denote by M_1 the point of intersection of $\bar{n}_M$ and $S_{t+\delta t}$, where $\delta t > 0$. We put

$$\overline{MM}_1 = \bar{\delta n} = \delta n \cdot \bar{n}^0, \tag{1.1.1}$$

where $\bar{n}^0$ is the unit vector normal n. Then $\delta n > 0$ if $M_1 \in D_{2,t}$, and $\delta n < 0$ if $M_1 \in D_{1,t}$. In other words, $\delta n > 0$ if in a neighborhood of the point M in the course of time δt a transformation from phase 2 to phase 1 occurs, accompanied by the liberation of latent heat; and, conversely, $\delta n < 0$ if the phase transition is accompanied by absorption of latent heat.

Let $d\sigma$ be an element of the surface S_t, containing a point M, and let $d\omega$ be a cylindrical element, constructed on $d\sigma$ as base, with the volume

$$|d\omega| = |\delta n|\, d\sigma. \tag{1.1.2}$$

We assume that the symbol $d\omega$ coincides with the known δn.

As a result of the phase transition, the formation of the phase in

the volume $d\omega$ involves an amount of heat equal to

$$\delta q = \lambda\rho d\omega = \lambda\rho d\sigma \cdot \delta n. \tag{1.1.3}$$

Relating δq to the unit time, we find that the value

$$\tilde{q} = \lambda\rho \frac{\delta n}{\delta t}, \tag{1.1.4}$$

equal to the rate of liberation of latent heat of the phase transition, may be regarded as the density of heat sources distributed over the surface of the interphase boundary. Hence it is obvious that on the interphase boundary S_t we have the condition

$$\left(k_1 \frac{\partial \vartheta_1}{\partial n} - k_2 \frac{\partial \vartheta_2}{\partial n}\right)_{S_t} = \lambda\rho \frac{\delta n}{\delta t}, \tag{1.1.5}$$

or in other words, the jump in the flow of heat as we pass through the transition boundary toward the side of the first phase is equal to the rate of discharge of latent heat of the phase transition. Condition (1.1.5) was derived by Stefan in the one-dimensional case in his work on freezing of the ground [167]. Below we call it the Stefan condition.

We will write the Stefan condition in an explicit form. Let (x,y,z) be a Cartesian coordinate system and let $F(x,y,z,t) = 0$ be the equation of the surface S_t. The normal equation of the tangent plane to S_t at M has the form

$$(X - x)\alpha + (Y - y)\beta + (Z - z)\gamma = 0, \tag{1.1.6}$$

where

$$\alpha = \frac{F_x}{\pm|\operatorname{grad} F|}; \quad \beta = \frac{F_y}{\pm|\operatorname{grad} F|}; \quad \gamma = \frac{F_z}{\pm|\operatorname{grad} F|}, \tag{1.1.7}$$

are the direction cosines of $\overline{\delta n}_M$ and X, Y, Z are running coordinates. If now (x_1,y_1,z_1) are the coordinates of the point M_1 and

$$dx = x_1 - x; \quad dy = y_1 - y; \quad dz = z_1 - z,$$

then

$$\delta n = \alpha dx + \beta dy + \gamma dz. \tag{1.1.8}$$

On the other hand,

$$F_x dx + F_y dy + F_z dz + F_t dt = 0. \tag{1.1.9}$$

Hence, from (1.1.7) and (1.1.8), we see that (1.1.5) can be written in the form

$$\lambda\rho\frac{\partial F}{\partial t}+(k_1\operatorname{grad}\vartheta_1-k_2\operatorname{grad}\vartheta_2;\operatorname{grad}F)_{S_t}=0. \tag{1.1.10}$$

§2. Freezing of the ground [167], [144]

The "ground" is a medium consisting of a rigid framework of particles and empty spaces filled with air and moisture under various effects of gravitation, capillarity and surface film. In each of these states the moisture is characterized by its temperature of crystallization. In order to investigate heat propagation in the ground and, in particular, to study the dynamics of its freezing, ground is necessarily considered as a multicomponent medium. We will assume that the particles of the ground are finely enough dispersed, so we can neglect the presence of gravitational moisture (i.e. moisture capable of moving under the influence of the force of gravity). In this case the transfer of heat in the ground by means of free convection is impossible and it can be assumed that the propagation of heat is the result of heat conduction alone.

Thus we consider a volume G of ground, and in it a small volume $\Delta\omega$. We assume that the number of particles contained in $\Delta\omega$ is very large, and that the dimensions of each of them are very small compared to $\Delta\omega$. Within $\Delta\omega$ there is a rigid framework of particles and in it moisture with crystallization temperatures $T_1,\cdots,T_n$. Let the relative volumes occupied by these particles be $p_0,\cdots,p_n$. We will speak of p_i as the "volume concentration of the particles of the ith component" ($i=0,\cdots,n$; the index $i=0$ refers to the particles of the frame itself). Assume that the particle size of each component is greater than molecular; let us assume in addition that the heat propagation within each component satisfies the Fourier law. At the same time we will assume that in the ground there is local thermodynamic equilibrium, so that the temperatures of all the particles in the individual components situated in the elementary volume $\Delta\omega$ are one and the same and equal to $\vartheta(x,y,z,t)$, where (x,y,z) are the Cartesian coordinates and t is the time. We will assume that the p_i ($i=0,\cdots,n$) are constants, and $\vartheta(x,y,z,t)$ is twice continuously differentiable with respect to the space coordinates and once with respect to the time, everywhere except at a finite number of smooth surfaces of discontinuity of the derivatives, these latter surfaces being isothermal. These surfaces will be interphase surfaces.

Let $k_{i,j}$, $c_{i,j}$ and $\rho_{i,j}$ be the coefficients of thermal conductivity, specific heat and density of particles of the ith component in the jth phase state ($i = 0, 1, 2, \cdots$; $j = 1, 2$). The index $j = 1$ will correspond to the solid phase, and $j = 2$ to the liquid. The change of volume concentration due to thermal expansion under a phase change will be neglected.

Assume that in the elementary volume $\Delta\omega$ the material of the ith component is in the jth phase state. Then the quantity

$$c = \sum_{m=0}^{n} c_{m,j_m} \rho_{m,j_m} p_m \tag{1.2.1}$$

is equal to the specific heat of the elementary volume $\Delta\omega$, referred to a unit volume.

Furthermore, let $d\sigma$ be an elementary element of area within $\Delta\omega$. This area intersects the part of the ith component in a set of two-dimensional measure $\mu_i d\sigma$. Under the above assumptions on the dimensions of the particles and under the assumption that they are randomly arranged within $\Delta\omega$, we may take $\mu_i = p_i$. Then

$$k = \sum_{m=0}^{n} k_{m,j_m} p_m \tag{1.2.2}$$

will be equal to the coefficient of thermal conductivity of the ground within the elementary volume $\Delta\omega$. Let

$$a^2 = \frac{k}{c}; \tag{1.2.3}$$

as stated above we will assume that the temperature distribution in $\Delta\omega$ obeys the Fourier equation with coefficient of temperature conductivity a^2.

Let Σ_t be the interphase surface of the ith component of the ground and let $D_{1,t}$ and $D_{2,t}$ be domains which have Σ_t as the common part of their boundaries and in which the material of the ith component is in the solid and the liquid phase, respectively. Denote by ϑ_j and k_j the temperature and coefficient of heat conductivity respectively of the ground in $D_{j,t}$ ($j = 1, 2$), and set λ_i equal to the specific latent heat of phase change of material in the ith physical state, and ρ_{i,j_i} equal to the density of the phase which is generated; then we see that the Stefan condition (1.1.10) can be written in the form

$$\lambda_i \rho_{i,j_i} p_i \frac{\partial F}{\partial t} + (k_1 \operatorname{grad} \vartheta_1 - k_2 \operatorname{grad} \vartheta_2, \operatorname{grad} F) = 0. \tag{1.2.4}$$

where $F = 0$ is the equation of the surface Σ_t.

In equation (1.2.4) for $\partial F/\partial t$ we introduce the factor $\lambda_i \rho_{i,j_i} p_i$ in place of the usual $\lambda\rho$, since the advance of the boundary Σ_t results from the crystallization of material in the ith component whose mass, within the basic volume $d\omega$, is equal to $\rho_{i,j_i} p_i d\omega$.

We will assume that the ground occupies the halfspace $x \geqq 0$. Moreover, we set

$$\vartheta(x,0) = 0; \quad \vartheta(0,t) = T_0 < T_1 < \cdots < T_n < 0, \tag{1.2.5}$$

where T_m is the crystallization temperature of the mth component.

Let the crystallization boundary of the material in the mth component be given by the function

$$x = y_m(t); \quad m = 1, 2, \cdots, n. \tag{1.2.6}$$

Moreover, let

$$y_0(t) = 0; \quad y_{n+1}(t) = \infty \tag{1.2.6*}$$

and denote by $\theta_m(x,t)$, k_m and a_m^2 the temperature, coefficient of thermal conductivity and thermal conductivity of the ground in the domain $y_{m-1}(t) < x < y_m(t)$ $(m = 1, \cdots, n+1)$. Let the crystallization temperatures T_i be arranged in order of magnitude

$$T_1 < T_2 < \cdots < T_n. \tag{1.2.7}$$

Then it is obvious that in the domain $y_{m-1}(t) < x < y_m(t)$ the particles of the ith phase are in the solid phase for $i = m, \cdots, n$ and in the liquid phase for $i = 1, \cdots, m-1$. Also, by (1.2.1) and (1.2.2), we have

$$\begin{aligned} k_m &= \sum_{i=1}^{m-1} k_{i,2} p_i + \sum_{i=m}^{n} k_{i,1} p_i; \\ c_m &= \sum_{i=1}^{m-1} c_{i,2} \rho_{i,2} p_i + \sum_{i=m}^{n} c_{i,1} \rho_{i,1} p_i; \\ a_m^2 &= \frac{k_m}{c_m}. \end{aligned} \tag{1.2.8}$$

The temperature distribution will be determined by the equation of heat conduction

$$a_m^2 \frac{\partial^2 \vartheta_m}{\partial x^2} = \frac{\partial \vartheta_m}{\partial t}, \quad y_{m-1}(t) < x < y_m(t), \quad t > 0, \tag{1.2.9$_1$}$$

the boundary conditions

$$\vartheta_1(0,t) = T_0; \quad \vartheta_m(y_m(t),t) = \vartheta_{m+1}(y_m(t),t) = T_m \quad (m = 1,2,\cdots,n), \tag{1.2.9$_2$}$$

the initial conditions

$$\vartheta_{n+1}(x,0) = 0, \quad 0 < x < \infty; \quad y_m(0) = 0 \quad (m = 1,2,\cdots,n) \tag{1.2.9$_3$}$$

and the Stefan conditions

$$\dot{y}_m(t) = \left(\alpha_m \frac{\partial \vartheta_m}{\partial x} - \beta_m \frac{\partial \vartheta_{m+1}}{\partial x}\right)\Bigg|_{x=y_m(t)} \quad (m = 1,2,\cdots,n). \tag{1.2.9$_4$}$$

Here

$$\alpha_m = \frac{k_m}{\lambda_m \rho_{m,1} p_m}; \quad \beta_m = \frac{k_{m+1}}{\lambda_m \rho_{m,2} p_m} \quad (m = 1,2,\cdots,n). \tag{1.2.9$_5$}$$

Problem (1.2.9$_1$)-(1.2.9$_4$) admits a similarity solution which can be written in the form

$$\begin{aligned} \vartheta_m(x,t) &= A_m + B_m \operatorname{erf} \frac{x}{2a_m\sqrt{t}}; \\ y_m(t) &= 2a_m c_m \sqrt{t} \quad (m = 1,2,\cdots,n). \end{aligned} \tag{1.2.10}$$

Thus equation (1.2.9$_1$) is satisfied automatically. Let the c_m be given and define A_m and B_m by the conditions (1.2.9$_2$) and (1.2.9$_3$), which give

$$A_{n+1} + B_{n+1} = 0; \quad A_1 = T_0; \tag{1.2.11}$$

$$A_m + B_m \operatorname{erf} c_m = A_{m+1} + B_{m+1} \operatorname{erf} \gamma_m c_m = T_m; \quad m = 1,2,\cdots,n.$$

Here

$$\gamma_m = \frac{a_m}{a_{m+1}}. \tag{1.2.11*}$$

The system (1.2.11) is easily solved. We find

$$A_1 = T_0; \quad B_1 = \frac{T_1 - T_0}{\operatorname{erf} c_1}; \tag{1.2.12}$$

. . . . . .

$$A_m = T_m - \frac{(T_m - T_{m-1}) \operatorname{erf} c_m}{\operatorname{erfc} c_m - \operatorname{erfc} c_{m-1}\gamma_{m-1}}; \quad B_m = \frac{T_m - T_{m-1}}{\operatorname{erf} c_m - \operatorname{erf} c_{m-1}\gamma_{m-1}};$$

$$m = 2,3,\cdots,n;$$

$$A_{n+1} = \frac{T_n}{\operatorname{erfc} \gamma_n c_n}; \quad B_{n+1} = \frac{T_n}{\operatorname{erfc} \gamma_n c_n}.$$

If we introduce (1.2.10) into the Stefan conditions (1.2.9$_4$) and take (1.2.12) into account, we obtain for the determination of c_m $(m = 1, \cdots, n)$ a system of transcendental equations

$$c_m = \frac{1}{a_m\sqrt{\pi}}\left[\frac{T_m - T_{m-1}}{\operatorname{erf} c_m - \operatorname{erf}\gamma_{m-1}c_{m-1}} \cdot \frac{\alpha_m}{a_m}\exp(-c_m^2)\right.$$

$$\left. - \frac{T_{m+1} - T_m}{\operatorname{erf} c_{m+1} - \operatorname{erf}\gamma_m c_m} \cdot \frac{\beta_m}{a_{m+1}}\exp(-\gamma_m^2 c_m^2)\right] \tag{1.2.13}$$

$$\equiv f_m(c_{m-1}, c_m, c_{m+1}); \quad (m = 1, 2, \cdots, n).$$

Here we take

$$c_0 = 0; \quad c_{n+1} = \infty; \quad T_{n+1} = 0. \tag{1.2.13*}$$

It is easy to prove the existence of a unique solution to the system (1.2.13) such that

$$0 < \gamma_m c_m < c_{m+1}, \quad m = 1, 2, \cdots, n. \tag{1.2.14}$$

In fact, choose an arbitrary $c_{n-1} \geqq 0$. Noting that $T_m - T_{m-1} > 0$, $(m = 1, \cdots, n+1)$, that $\operatorname{erf} c_{n+1} - \operatorname{erf}\gamma_n c_n = \operatorname{erfc}\gamma_n c_n > 0$, and that γ_m, α_m, β_m, $a_m > 0$, we find that

$$f_n(c_{n-1}, c_n, c_{n+1}) < 0 \quad \text{for} \quad c_n < \gamma_{n-1}c_{n-1}. \tag{1.2.15}$$

Furthermore, for $c_n > \gamma_{n-1}c_{n-1}$ and c_{n-1} constants, $f_n(c_{n-1}, c_n, c_{n+1})$ varies monotonically from $+\infty$ for $c_n = \gamma_{n-1}c_{n-1}$ to $-\infty$ for $c_n = \infty$. Consequently (1.2.13) uniquely defines $c_n = c_n(c_{n-1})$, and (1.2.14) is in fact realized for $m = n - 1$.

Let us now assume that the uniquely defined function $c_k = c_k(c_{k-1})$ $(k = m+1, \cdots, n)$ satisfies (1.2.14). In this case

$$f_m(c_{m-1}, c_m, c_{m+1}) = f_m(c_{m-1}, c_m, c_{m+1}(c_m)) \equiv \phi_m(c_{m-1}, c_m). \tag{1.2.16}$$

Let us fix $c_{m-1} \geqq 0$. From (1.2.14) we have $\phi_m(c_{m-1}, c_m) < 0$ for $c_m < \gamma_{m-1}c_{m-1}$, and thus the inequality $c_m \geqq 0$ is impossible. If $c_m > \gamma_{m-1}c_{m-1}$, then for fixed c_{m-1}, $\phi_m(c_{m-1}, c_m)$ varies monotonically from $+\infty$ for $c_m = \gamma_{m-1}c_{m-1} + 0$ to $-\infty$ for $c_m \to \infty$. Hence

$$\frac{\exp(-\gamma_m^2 c_m^2)}{\operatorname{erf} c_{m+1} - \operatorname{erf}\gamma_m c_m} > \frac{\exp(-\gamma_m^2 c_m^2)}{\operatorname{erfc}\gamma_m c_m}; \quad \lim_{z\to\infty}\frac{\exp(-z^2)}{\operatorname{erfc} z} = \infty. \tag{1.2.17}$$

Thus the equation $c_m = \phi(c_{m-1}, c_m)$ uniquely defines $c_m = c_m(c_{m-1})$. If we continue the function linearly from $m = 1$ and note that, according to (1.2.13*), $c_0 = 0$, we have proved our assertion.

§3. Crystallization of a melt when a plate is immersed in it [16], [3]

We consider a melt maintained in a state of intensive mixing at temperature T_1. At the moment $t = 0$ an unbounded plate of thickness $2l$ with temperature $T_0 < T_1$ is immersed in this melt. We assume that a known heat flux $q = \text{const}$ is maintained from within the melt to the body immersed in it, and that at the boundary layer the temperature changes linearly in the direction normal to the surface of the plate. The problem is to follow the process of change of phase state in the melt and the plate.

Let T_k be the crystallization temperature of the melt and T_n the melting temperature of the plate. Let us assume at first that

$$T_0 < T_k < T_1 < T_n. \tag{1.3.1}$$

In this case the immersion of the plate in the melt causes the formation on the plate of a crust, the thickness of which grows until some time $t_0 > 0$, after which it begins to melt. The process of melting of the crust ends at some moment $t_1 > t_0$, when the temperature of the surface of the plate becomes equal to T_k. Subsequently there occurs a heating of the plate without phase transition. (We assume that the plate is insoluble in the material of the melt.) We will not consider this stage of the process.

If

$$T_0 < T_k < T_n < T_1, \tag{1.3.2}$$

then there is a time $t_2 > t_1$ such that for $t = t_2$ the temperature of the plate is T_n. Assume that the process of phase change takes place without superheating or supercooling; we find in this case that the plate begins to melt at time t_2. This process will last until the time t_3 at which the complete melting of the plate is attained.

We consider, finally, the case

$$T_0 < T_n < T_k < T_1, \tag{1.3.3}$$

In this case the state of the conducting medium will at some time be in four phases; namely, the material of the plate will be in the solid, as well as in the liquid phase, and there will be a solid crust on the plate, crystallized from the melt. However, it is not known beforehand what will be the initial stage of the process.

A priori we have three possibilities:

a) Melting of the plate and crystallization of the melt starting instantly at the moment of immersion of the plate in the melt.

b) Crystallization of the melt starting instantly, and melting of the plate at some time after its immersion.

c) Melting of the plate beginning instantly, and crystallization of the melt with some delay.

A strict analysis of these possibilities and determination of the real character of the initial state of the process will be carried out in Part 2 (see Chapter V, §5). Here we still restrict ourselves to the formulation and approximate solution of the problem under the assumption that (1.3.1) or (1.3.3), referred to as case "b," holds.

Thus, we assume condition (1.3.1). We assign the index $i = 1$ to the liquid and $i = 2$ to the solid phase crystallizing from the plate respectively. Let T, k_i, c_i, ρ_i and $a_i^2 = k_i/c_i\rho_i$ be the temperature, conductively coefficient, specific heat, density and coefficient of thermal diffusivity of the ith phase. We assume that k_i, c_i and ρ_i are constant. The difference between the density of the plate and solid crust (like the difference of the densities of the solid and liquid phases of the material of the plate in case (1.3.3)) are neglected. Finally, we will denote by γ the specific latent heat of crystallization of the melt and by q = constant the heat flux from within the melt to the solid phase, per unit area.

We will use an x, y, z Cartesian coordinate system with the plate initially parallel to the y, z plane, symmetric with respect to it and normal to the x-axis. Let t be the time, counted from the moment of immersion of the plate into the melt, and let $x = X(t)$ be the equation of the boundary of crystallization of the melt. With these conditions the problem consists of determining the functions $T(x,t)$ and $X(t)$ from the conditions

$$(1.3.4_1) \qquad a^2(x)\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}, \quad 0 < x < X(t), \quad x \neq l, \quad t > 0,$$

$$(1.3.4_2) \qquad \left.\frac{\partial T}{\partial x}\right|_{x=+0} = 0; \quad T|_{x=X(t)-0} = T_k; \quad T|_{t=+0} = T_0$$

$$(1.3.4_3) \qquad T|_{x=l-0} = T|_{x=l+0}; \quad k_1\left.\frac{\partial T}{\partial x}\right|_{x=l-0} = k_2\left.\frac{\partial T}{\partial x}\right|_{x=l+0},$$

$$(1.3.4_4) \qquad \gamma\rho_2\dot{X}(t) = k_2\left.\frac{\partial T}{\partial x}\right|_{x=X(t)} - q; \quad X(0) = l.$$

Here

$$(1.3.4_5)\quad a^2 = \begin{cases} a_1^2 & \text{for } 0 < x < l; \\ a_2^2 & \text{for } l < x < X(t); \end{cases} \qquad T = \begin{cases} T^1 & \text{for } 0 < x < l; \\ T^2 & \text{for } l < x < X(t). \end{cases}$$

Conditions $(1.3.4_4)$ follow at once from the Stefan conditions and from the assumption of linear termperature distribution in the liquid boundary layer.

We will sketch the method of the approximation of the solution to problem $(1.3.4_i)$, $(i = 1, \cdots, 5)$, presented in [3] on the basis of the method of L. S. Leĭbenzon [92]. If we integrate $(1.3.4_1)$ with respect to x from 0 to l and use the first of conditions $(1.3.4_2)$ and the second of conditions $(1.3.4_3)$, we find that

$$(1.3.5)\qquad k_2 \frac{\partial T}{\partial x}\bigg|_{x=l+0} = c_1 \rho_1 \frac{\partial}{\partial t} \int_0^l T(x,t)\, dx.$$

Following Leĭbenzon, we now consider the temperature in each phase to be quasi-stationary. This means that $T(x,t)$ is piecewise linear with respect to x and has a corner point for $x = l$. But it follows from conditions $(1.3.4_2)$ that

$$(1.3.6)\qquad \frac{\partial T}{\partial x}\bigg|_{x<l} \equiv 0; \quad \frac{\partial T}{\partial x}\bigg|_{x>l} = \frac{T_k - T(l,t)}{X(t) - l}.$$

We introduce the notation

$$u = \frac{T(l,t) - T_0}{T_k - T_0}; \quad \delta = \frac{X(t) - l}{l};$$

$$(1.3.7)\qquad \alpha = AC = \frac{c_1 \rho_1 l q}{k_2 \rho_2 \gamma}; \quad \beta = \frac{B}{C} = \frac{c_1 \rho_1 (T_k - T_0)\, l}{q};$$

$$A = \frac{c_1 \rho_1 l^2}{k_2}; \quad B = \frac{c_1 \rho_1 (T_k - T_0)}{\gamma \rho_2}; \quad C = \frac{q}{\gamma \rho_2 l}.$$

In this notation conditions $(1.3.4_4)$ and (1.3.5) can be written in the form

$$(1.3.8)\qquad A \frac{du}{dt} = \frac{1 - u}{\delta}; \quad \frac{d\delta}{dt} = B \frac{du}{dt} - C.$$

The system (1.3.8) can be integrated, with the initial conditions

$$(1.3.9)\qquad u|_{t=0} = 0; \quad \delta|_{t=0} = 0.$$

This gives

$$\delta = Bu - Ct; \quad t = \beta\left[u + \frac{1-u}{1-\alpha} - \frac{(1-u)^{\alpha}}{1-\alpha}\right]. \tag{1.3.10}$$

The moment t_k of termination of the melting of the crust is found from the condition $u = 1$, which gives

$$t_k = \beta. \tag{1.3.11}$$

The maximum thickness of the crust δ_m and the corresponding time t_m are reached for $u = u_m$, determined from the condition

$$\frac{d\delta}{du} = 0, \quad 0 < u_m < 1.$$

Differentiating (1.3.10) yields

$$u_m = 1 - \alpha^{1/(1-\alpha)} \tag{1.3.12}$$

whence

$$t_m = \beta[1 - (1+\alpha)\alpha^{\alpha/(1-\alpha)}]; \quad \delta_m = B\alpha^{\alpha/(1-\alpha)}. \tag{1.3.13}$$

Suppose that the process is of type "b" in which condition (1.3.3) is realized. In this case the initial stage of the process is described by relations $(1.3.4_1)$—$(1.3.4_5)$, and consequently with the approximation adopted we are led to equation (1.3.10).

Let $t_{\text{п}}$ be the intial moment of melting of the plate, δ_n the thickness of the crust at the time $t_{\text{п}}$, and $t_k, \tilde{t}_k$ the final times of melting of the crust and of the plate respectively. Then

$$t_{\text{п}} = t(u_{\text{п}}); \quad \delta_{\text{п}} = \delta(u_{\text{п}}), \tag{1.3.14}$$

where $u_{\text{п}}$ is the dimensionless temperature of the melted plate

$$u_{\text{п}} = \frac{T_{\text{п}} - T_0}{T_k - T_0}. \tag{1.3.14*}$$

We shall write out the equations of the problem on the interval $t_{\text{п}} < t < \min(t_k, \tilde{t}_k)$,[1)] again neglecting variations of the density under phase change, as well as the formation of convection currents in the liquid phase formed from the melt of the plate. Then we must have

[1)] The magnitudes of t_k and $\tilde{t}_k$ are calculated under the assumption that the complete melting of the crust or of the plate precedes the complete melting of the plate or of the crust, respectively. This t_k obviously does not coincide with t_k as defined earlier.

$$(1.3.15_1) \qquad b^2(x)\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}, \quad 0 < x < X(t); \quad x \neq \widetilde{X}(t), \quad x \neq l, \quad t > t_{\text{п}},$$

$$(1.3.15_2) \qquad \left.\frac{\partial T}{\partial x}\right|_{x=+0} = 0; \quad T|_{x=\widetilde{X}-0} = T|_{x=\widetilde{X}+0} = T_{\text{п}}; \quad T|_{x=X-0} = T_k,$$

$$(1.3.15_3) \qquad T|_{x=l-0} = T|_{x=l+0}; \quad \tilde{k}_1 \frac{\partial}{\partial x} T|_{x=l-0} = k_2 \frac{\partial}{\partial x} T|_{x=l+0},$$

$$(1.3.15_4) \qquad \tilde{\gamma}\rho_1 \dot{\widetilde{X}} = k_1 \frac{\partial}{\partial x} T|_{x=\widetilde{X}-0} - \tilde{k}_1 \frac{\partial}{\partial x} T|_{x=\widetilde{X}+0}; \quad \widetilde{X}(t_{\text{п}}) = l,$$

$$(1.3.15_5) \qquad \gamma\rho_2 \dot{X} = k_2 \frac{\partial}{\partial x} T|_{x=X-0} - q; \quad X(t_{\text{п}}) = X_{\text{п}} > l, \quad t > t_{\text{п}}.$$

Here

$$(1.3.15_6) \qquad b^2(x) = \begin{cases} a_1^2 = \dfrac{k_1}{c_1\rho_1}, & 0 < x < \widetilde{X}(t); \\ \tilde{a}_1^2 = \dfrac{\tilde{k}_1}{\tilde{c}_1\rho_1}, & \widetilde{X}(t) < x < l; \quad X_{\text{п}} = l(1+\delta_{\text{п}}). \\ a_2^2 = \dfrac{k_2}{c_2\rho_2}, & l < x < X(t); \end{cases}$$

In the quasi-stationary approximation

$$(1.3.16) \qquad T(x,t) = \begin{cases} T_{\text{п}}, & 0 < x < \widetilde{X}(t); \\ T_{\text{п}} + \dfrac{T(l,t) - T_{\text{п}}}{l - \widetilde{X}(t)} \cdot (x - \widetilde{X}(t)), & \widetilde{X} < x < l; \\ T(l,t) + \dfrac{T_k - T(l,t)}{X(t) - l}(x - l), & l < x < X(t). \end{cases}$$

Conditions $(1.3.15_4)$ and $(1.3.15_5)$ and the last of conditions $(1.3.15_3)$ are now written in the form

$$\tilde{k}_1 \frac{T(l,t) - T_{\text{п}}}{l - \widetilde{X}(t)} = k_2 \frac{T_k - T(l,t)}{X(t) - l};$$

$$(1.3.17) \qquad \tilde{\gamma}\rho_1 \dot{\widetilde{X}} = -\tilde{k}_1 \frac{T(l,t) - T_{\text{п}}}{l - \widetilde{X}(t)}; \qquad X|_{t=t_{\text{п}}} = l;$$

$$\gamma\rho_2 \dot{X} = +k_2 \frac{T_k - T(l,t)}{X(t) - l} - q; \qquad X|_{t=t_{\text{п}}} = X_{\text{п}}.$$

Integrating (1.3.17), taking (1.3.7) into account and using the notation

$$\delta_1 = \frac{l - \widetilde{X}(t)}{l}; \qquad \alpha_1 = l \cdot \frac{k_2\gamma\rho_2 + \widetilde{k}_1\widetilde{\gamma}\rho_1}{\widetilde{k}_1 q};$$

$$\beta_1 = \frac{\widetilde{\gamma}\rho_1 lq}{\gamma\rho_2 k_2(T_k - T_\text{п})}; \qquad \alpha_2 = \frac{\widetilde{\gamma}\rho_1 l^2\delta_\text{п}}{k_2(T_k - T_\text{п})}; \tag{1.3.18}$$

$$\gamma_1 = \alpha_2 - \alpha_1,$$

we obtain

$$t - t_\text{п} = \alpha_1\delta_1 + \frac{\gamma_1}{\beta_1}[1 - \exp(-\beta_1\delta_1)];$$

$$\delta - \delta_\text{п} = \frac{\widetilde{\gamma}\rho_1}{\gamma\rho_2}\delta_1 - \frac{q}{l\gamma\rho_2}(t - t_\text{п}); \qquad u - u_\text{п} = \frac{k_2(1 - u_\text{п})\,\delta_1}{\widetilde{k}_1\delta + k_2\delta_1}. \tag{1.3.19}$$

We define t_k and $\widetilde{t}_k$ by the equations

$$t_k = t|_{\delta=0}; \qquad \widetilde{t}_k = t|_{\delta_1=1}. \tag{1.3.20}$$

Then (1.3.19) defines the process for

$$t_\text{п} \leqq t \leqq t_M = \min(t_k, \widetilde{t}_k). \tag{1.3.21}$$

If $t_k < \widetilde{t}_k$, then t_k coincides with the time of total melting of the crust. In this case, at the moment t_k there occurs immediate intermixing of the liquid phase, formed by melting of the plate, with the body of the melt; the subsequent process will be a one-phase process. This will be described by the equations

$$a_1^2\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}, \qquad 0 < x < \widetilde{X}(t), \quad t > t_M;$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=+0} = 0; \qquad T|_{x=\widetilde{X}(t)} = T_\text{п}; \tag{1.3.22}$$

$$\widetilde{\gamma}\rho_1\dot{\widetilde{X}} = k_1\left.\frac{\partial T}{\partial x}\right|_{x=\widetilde{X}(t)} - q; \qquad \widetilde{X}|_{t=t_M} = \widetilde{l} = l(1 - \delta_1(t_k)).$$

Consider the approximation

$$T|_{t=t_M+0} = T_\text{п}. \tag{1.3.23}$$

From this and from the two conditions (1.3.22) it is evident that

$$T \equiv T_\text{п}, \quad t > t_M. \tag{1.3.24}$$

Integrating the last of the conditions (1.3.22), we find that therefore

$$\widetilde{X}(t) = \widetilde{l} - q(t - t_M)/\gamma\rho_1. \tag{1.3.25}$$

Setting $\widetilde{X}(t) = 0$, we determine the moment $\widetilde{t}_k$ of total melting:

$$\widetilde{t}_k = t_M + \frac{\widetilde{\gamma}\rho_1 \widetilde{l}}{q}, \quad t_M = t_k. \tag{1.3.26}$$

Noting that the second equation of (1.3.19) implies

$$t_k - t_\Pi = \frac{l}{q}\left[\gamma\rho_2\delta_\Pi + \widetilde{\gamma}\rho_1\delta_1(t_k)\right], \tag{1.3.27}$$

we find that (1.3.26) is equivalent to the equation

$$\widetilde{t}_k - t_\Pi = \frac{l}{q}\left[\widetilde{\gamma}\rho_1 + \gamma\rho_2\delta_\Pi\right]. \tag{1.3.28}$$

Thus in the quasi-stationary approximation the time of complete melting of the plate is found to be independent of the time of complete melting of the crust, provided, of course, that $t_k < \widetilde{t}_k$.

Assume now that $t_M = \widetilde{t}_k$. In this case the plate will be entirely melted before the end of the melting of the crust. Subsequently the course of the process will be described by the equations

$$\widetilde{a}_1^2 \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}, \quad 0 < x < l; \qquad a_2^2 \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}, \quad l < x < X(t), \quad t > \widetilde{t}_k;$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0; \quad T|_{x=l-0} = T|_{x=l+0}; \quad \widetilde{k}_1 \left.\frac{\partial T}{\partial x}\right|_{x=l-0} = k_2 \left.\frac{\partial T}{\partial x}\right|_{x=l+0}; \tag{1.3.29}$$

$$T|_{x=X(t)-0} = T_k;$$

$$\gamma\rho_2 \dot{X} = k_2 \left.\frac{\partial T}{\partial x}\right|_{x=X(t)} - q; \qquad X|_{t=t_k} = X_M,$$

where X_M is defined by the condition

$$X_M = l(1 + \delta(\widetilde{t}_k)). \tag{1.3.30}$$

Here it is necessary to add the initial condition

$$T|_{t=t_M} = T_M, \tag{1.3.31}$$

defined in our approximate treatment of the equations (1.3.7), (1.3.16), (1.3.18) and (1.3.19) for $t = t_M$.

We replace c_1 by $\widetilde{c}_1$ in (1.3.7) and write $\widetilde{A}, \widetilde{B}, \widetilde{\alpha}$ and $\widetilde{\beta}$ in place of A, B, α and β. It is easily seen that in the quasi-stationary approximation

$$\widetilde{A}\frac{du}{dt} = \frac{1-u}{\delta}; \quad \frac{d\delta}{dt} = \widetilde{B}\frac{du}{dt} - C; \quad u|_{t=t_M} = u_M; \quad \delta|_{t=t_M} = \delta_M. \tag{1.3.32}$$

Integration gives

$$\delta - \delta_M = \widetilde{B}(u - u_M) - C(t - t_M),$$

$$t - t_M = \widetilde{\beta}(u - u_M) + \frac{\delta_M}{C}\left[1 - \left(\frac{1-u}{1-u_M}\right)^{\widetilde{\alpha}}\right] \tag{1.3.33}$$

$$+ \frac{\widetilde{\beta}}{1-\widetilde{\alpha}}\left[1 - u - (1-u_M)\left(\frac{1-u}{1-u_M}\right)^{\widetilde{\alpha}}\right].$$

In Supplement 3, §2, we describe the numerical calculation resulting from this method; in particular, we compare the results of approximation of the solution of the problem with its exact solution. This comparison, in the case $T_n > T_k$, shows that the quasi-stationary approximation gives a qualitatively valid and quantitatively satisfactory picture of the process (in a broad range of the varying parameters).

In the case $T_n < T_k$ the calculation has not been performed. We may, however, note that in the last of the cases examined above the quasi-stationary approximation automatically gives a distorted result, since in this approximation, at the moment of total melting of the slab, the slope of the line describing the dependence of temperature on x immediately changes. In exactly the same way we obtain qualitatively unsatisfactory results from (1.3.28), about the dependence of $\widetilde{t}_k$ on t_k in the case $t_k < \widetilde{t}_k$; it is thus necessary to examine the process with more exactness than is possible by the quasi-stationary treatment.

§4. Formation of a continuous ingot [180]

The process of formation of a continuous ingot consists of the following: Given a casting form, consisting of a crystallizer mold A and a mobile platform B, moving at the initial moment into contact with the lower part of the mold. The interior wall of the mold is cooled by a continuous stream of water. Also the interior of the mold is filled with molten metal up to the level O—D. As soon as a sufficiently stable core is formed, the platform B begins to drop down and thus draws out a continuous cast of cooling liquid (see Figure 2). The problem consists of determining asymtotically for $t \to \infty$ the temperature distribution and the form and position of the stationary front of crystallization. We make the following assumptions:

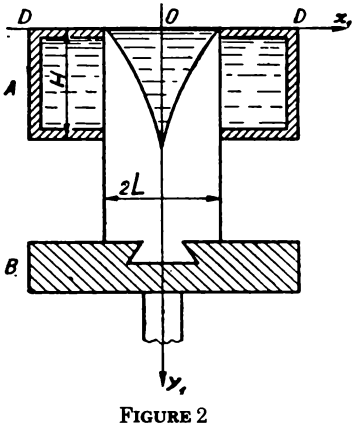

FIGURE 2

a) The level of the metal in the mold is maintained constant, as is the temperature of the metal at this level.

b) Between the exterior surface of the ingot at the mold and the cooling shell, there is a heat exchange given by Newton's law with constant heat exchange coefficent.

c) Outside the mold the temperature at the surface of the ingot is constant and equal to the temperature of the water in the casing of the mold.

d) The temperature of the liquid phase differs from the temperature of crystallization by a negligible amount.

In agreement with the last assumption the temperature of the liquid phase is taken as constant, and equal to the crystallization temperature of the ingot.

We will limit our discussion to the case of the formation of a plane ingot with unbounded width. Let $v = \text{const}$ be the velocity at which the ingot is drawn out, t_1 the time, counted from the initial drawing out of the cast, and (x_1, y_1) a point fixed to the mold such that for $t_1 \to \infty$ the domain occupied by the ingot will be defined by the inequality $-L < x_1 < L$; $0 < y_1 < \infty$. Let us introduce a moving coordinate system (ξ_1, μ_1) fixed to the ingot such that

$$\xi_1 = x_1; \qquad \eta_1 = y_1 - vt_1, \tag{1.4.1}$$

and the dimensionless variables

$$x = \frac{x_1}{L}; \quad y = \frac{y_1}{L}; \quad \xi = \frac{\xi_1}{L}; \quad \eta = \frac{\eta_1}{L} \equiv y - \frac{t}{\mu};$$

$$t = \frac{a^2}{L^2} t_1; \quad \mu = \frac{a^2}{vL}; \quad u = \frac{T - T_0}{T_1 - T_0}; \quad \lambda = \frac{\gamma d v L}{k(T_0 - T_1)} \tag{1.4.2}$$

$$\nu = \lambda\mu; \quad \alpha = \frac{\beta L}{k}; \quad l = \frac{H}{L},$$

where a^2 and k are the coefficients of diffusivity and conductivity; β is the coefficient of heat transfer; d is the density; γ is the specific latent heat of crystallization; T_0 is the temperature of crystallization; T_1 is the temperature of the water in the mold and also of the surface of the ingot outside the mold; H is the depth of the mold and $2L$ is its thickness.

Since the $x = 0$ axis is an axis of symmmetry, it is sufficient to consider the domain $0 < x < 1$, $0 < y$ only. In the moving coordinate system the temperature of the ingot is determined by the conditions

$$\frac{\partial^2 u}{\partial \xi^2}+\frac{\partial^2 u}{\partial \eta^2}=\frac{\partial u}{\partial t}, \quad f(\eta,t)<\xi<1, \quad 0<y\equiv\eta+\frac{t}{\mu}<\infty, \quad t>0;$$

(1.4.3)
$$\left.\frac{\partial u}{\partial \xi}\right|_{\xi=1}=\alpha(1-u)|_{\xi=1} \quad \text{for} \quad 0<y<l;$$

$$u|_{\xi=1}=1 \quad \text{for} \quad y>l; \quad t>0; \quad u|_{\xi=f(\eta,t)}=0.$$

Here $\xi=f(\eta,t)$ is the equation of the crystallization front. We note that the length l_0 of the front is determined by the equation $f(\eta,t)=0$. For $\eta+t/\mu>l_0$ the first of the equations (1.4.3) will be realized for $0<\xi<1$.

By virtue of equation (1.4.2) and the assumption on the constancy of temperature of the liquid phase, the Stefan condition (1.1.10) can be written in the form

(1.4.4)
$$\nu\frac{\partial f}{\partial t}+\left(\frac{\partial u}{\partial \xi}-\frac{\partial f}{\partial \eta}\frac{\partial u}{\partial \eta}\right)\Bigg|_{\xi=f(\eta,t)}=0.$$

From assumption a) it evidently follows that

(1.4.4*)
$$f(\eta,t)=1 \quad \text{for} \quad y=\eta+\frac{t}{\mu}=0.$$

The initial condition is not written, since we are only interested in the asymptotic temperature distribution for $t\to\infty$. For its determination we turn to equations of the problem in stationary coordinates (x,y) and in these equations we set the derivatives with respect to t equal to zero. We obtain

(1.4.5$_1$)
$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{1}{\mu}\frac{\partial u}{\partial y}, \quad \phi(y)<x<1 \quad \text{for} \quad 0<y<l_0,$$

(1.4.5$_2$)
$$\left.\frac{\partial u}{\partial x}\right|_{x=1}=\alpha(1-u)|_{x=1} \quad \text{for} \quad 0<y<l; \quad u|_{x=1}=1 \quad \text{for} \quad y>l,$$

(1.4.5$_3$)
$$u|_{x=\phi(y)}=0 \quad \text{for} \quad y<l_0; \quad \left.\frac{\partial u}{\partial x}\right|_{x=0}=0 \quad \text{for} \quad y>l_0,$$

(1.4.5$_4$)
$$-\lambda\dot{\phi}(y)=\left(\frac{\partial u}{\partial x}-\dot{\phi}(y)\frac{\partial u}{\partial y}\right)\Bigg|_{x=\phi(y)}; \quad \phi(0)=1.$$

Here

$$\phi(y)=\lim_{t\to\infty} f\left(y-\frac{t}{\mu},t\right); \quad \phi(y)\equiv 0 \quad \text{for} \quad y>l_0; \quad \phi(l_0)=0.$$

The equation

$$x = \phi(y), \quad 0 < y < l_0 \tag{1.4.5$_5$}$$

is the equation of the unknown stationary front of crystallization.

Note that (1.4.5$_1$) has the form of a steady state equation of heat conduction with forced convection, where the role of the convective term is played by $\mu^{-1}\,\partial u/\partial y$.

Assume now that the velocity at which the ingot is withdrawn is great enough that the transfer of heat by conduction in the direction of the y-axis can be ignored in comparison to the convection transfer of heat in this direction. Thus equation (1.4.5$_1$) can be replaced by

$$\mu \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y}, \quad \phi(y) < x < 1, \quad 0 < y, \tag{1.4.5$_1'$}$$

a formula coinciding with the equation of heat conduction, in which y plays the role of time, and μ is the coefficient of thermal diffusivity.

We note, further, that in the Stefan condition (1.4.5$_4$) the term $-\lambda\dot{\phi}(y)$ expresses the loss of the latent heat of crystallization from the moving ingot, and the term $-\dot{\phi}(y)\,\partial u/\partial y$ expresses the transfer of heat in the direction of the y-axis by conduction. Under our assumptions it is possible to neglect this last term. Thus the Stefan condition is approximated by a relation of the form

$$-\lambda\dot{\phi}(y) = \left.\frac{\partial u}{\partial x}\right|_{x=\phi(y)}; \quad \phi(0) = 1.$$

The solution is therefore to be determined from equations (1.4.5$_1$), (1.4.5$_2$), (1.4.5$_3$), (1.4.5$_4'$). We give an approximate solution of this problem for a value of y varying from 0 to l, i.e. in the bounds of the mold. We will use the quasi-stationary approximation of Leĭbenzon: namely, we approximate the temperature by a linear function of x satisfying the boundary conditions of the problem and depending on y as if on a parameter:

$$u(x, y) = -\frac{\alpha(\phi - x)}{1 + \alpha(1 - \phi)}. \tag{1.4.6}$$

If we introduce (1.4.6) into the Stefan condition (1.4.5$_4'$), we obtain the differential equation for the determination of ϕ:

$$-\lambda\dot{\phi} = \frac{\alpha}{1 + \alpha(1 - \phi)}; \quad \phi(0) = 1. \tag{1.4.7}$$

Setting

$$\delta = 1 - \phi, \tag{1.4.8}$$

we find that

$$y = \frac{\lambda}{\alpha}\left(\delta + \frac{\alpha}{2}\delta^2\right). \tag{1.4.9}$$

We obtain another approximation if we use the integral form of the condition of solidification as done by Leĭbenzon [92]. Namely, integrating (1.4.5$_4'$) with respect to x and taking into account the boundary conditions of the problem, we obtain

$$\lambda\dot{\phi}(y) = -\alpha(1-u)_{x=1} + \frac{1}{\mu}\int_{\phi(y)}^{1}\frac{\partial u}{\partial y}dx. \tag{1.4.10}$$

Substituting (1.4.6) and (1.4.8) into (1.4.10) results in the equation

$$\alpha = \frac{1}{2\mu}\left[(2\lambda\mu+1)(1+\alpha\delta) - \frac{1}{1+\alpha\delta}\right]\dot{\delta};\quad \delta(0) = 0. \tag{1.4.11}$$

Integration yields

$$y = \frac{1}{2\alpha^2\mu}\left[\alpha(2\lambda\mu+1)\left(\delta + \frac{\alpha\delta^2}{2}\right) - \ln(1+\delta\alpha)\right]. \tag{1.4.12}$$

In order that we obtain the expression $\phi(y)$ outside of the mold, it is sufficient in our framework to consider approximations of equations (1.4.7) and (1.4.11) in the limit for $\alpha \to \infty$ and to integrate them with regard to the initial condition

$$\phi(l) = \phi_0. \tag{1.4.13}$$

Here ϕ_0 is found from (1.4.9) and correspondingly (1.4.12), if in it we take $y = l$ and $\delta_0 = 1 - \phi_0$. Thus we obtain two approximations for y:

$$y = l + \frac{\lambda}{2}(\delta^2 - \delta_0^2),\quad l = \frac{\lambda}{\alpha}\left(\delta_0 + \frac{\alpha}{2}\delta_0^2\right); \tag{1.4.14}$$

$$\begin{aligned} y &= l + \frac{2\lambda\mu+1}{4\mu}(\delta^2 - \delta_0^2), \\ l &= \frac{1}{2\alpha^2\mu}\left[\alpha(2\lambda\mu+1)\left(\delta_0 + \frac{\alpha\delta_0^2}{2}\right) - \ln(1+\alpha\delta_0)\right]. \end{aligned} \tag{1.4.15}$$

It is possible to show that the expressions (1.4.9), (1.4.14) and (1.4.12), (1.4.15) approximate $\phi = 1 - \delta$ from above and from below respectively.

We note that for small values of α both approximations practically coincide with the functions

$$\delta = \frac{\alpha}{\lambda} y \quad \text{and} \quad \delta = \frac{2\alpha\mu}{2\lambda\mu + 1} y, \quad \text{respectively.} \tag{1.4.16}$$

§5. Zonal noncrucible melting of a cylindrical rod

In conclusion, we formulate the problem of zonal noncrucible melting of a cylindrical rod, which is similar to the problem considered in §4 for the formation of a continuous ingot, but which, in contrast to that problem, does not admit reduction to a one-dimensional treatment.

We consider a cylindrical rod, moving forward with constant speed in the field of a high-frequency inductor. In the domain of influence of this induction, the material rod melts and retains its form by means of surface tension and magnetic attraction. On the surface of the rod there is radiation according to the Stefan-Boltzmann principle.

It is required to determine, asymptotically with respect to time, the position of the isotherm of melt and crystallization, and the dependence of the position and form of this isotherm on parameters characterizing the work of the inductor and on the thermal-physical characteristics of the rod.

We assume that

a) The energy radiating from the inductor is absorbed at the surface of the rod and can be considered as defining a continuously distributed heat source on its surface.

b) The difference between the boundary of the rod and the domain of melting on the initial cylindrical surface can be neglected.

c) The temperature distribution within the liquid phase can be characterized by the Fourier equation with a known effective value for the heat conduction coefficient.

Let R_0 be the radius of the rod, v the velocity at which it moves, and $\Phi(x)$ the density of the energy flux radiating from the inductor per unit surface of the rod. Here the x-axis is the stationary axis, linked to the inductor and coincident with the axis of the rod. Thus we assume that the energy flow radiating from the inductor has axial symmetry. Moreover, we will assume an axially symmetric distribution of temperature, and that $\Phi(x)$ is symmetric with respect to the plane $x = 0$.

We introduce a moving coordinate system, fixed to the rod, letting

$$z = x - vt, \tag{1.5.1}$$

where t is the time, measured from the initial moment of movement

of the rod. We regard the coefficients of heat conduction and temperature conductivity of the rod as piecewise constant functions of the temperature with discontinuities for the critical temperature

$$(1.5.2) \qquad a^2(T) = \begin{cases} a_1^2, T < T_{\text{п}} \\ a_2^2, T > T_{\text{п}} \end{cases}, \quad k(T) = \begin{cases} k_1, T < T_{\text{п}} \\ k_2, T > T_{\text{п}} \end{cases};$$

neglecting thermal expansion of the material of the rod and the variable density ρ for the phase change, we find that the temperature distribution in the rod obeys the following conditions:

$$(1.5.3_1) \qquad a^2(T)\,\Delta T = \frac{\partial T}{\partial t}, \quad 0 < r < R_0, \quad -\infty < z < \infty; \quad T \neq T_{\text{п}}; \quad t > 0,$$

$$(1.5.3_2) \qquad -k(T)\frac{\partial T}{\partial r}\bigg|_{r=R_0} = \sigma T^4|_{r=R_0} - \Phi(z + vt), \quad t > 0,$$

$$(1.5.3_3) \qquad \lambda\rho\frac{\partial F}{\partial t} + (k_1 \operatorname{grad} T|_{T=T_{\text{п}}-0} - k_2 \operatorname{grad} T|_{T=T_{\text{п}}+0}, \operatorname{grad} F) = 0,$$

where σ is the Stefan-Boltzmann constant, λ is the specific latent heat of melting and $F = 0$ is the equation of the isotherm $T = T_{\text{п}}$.

Moreover, at infinity the equation

$$(1.5.3_4) \qquad \lim_{|z|\to\infty} T = T_0 < T_{\text{п}}$$

will be obeyed. We do not write out the initial conditions, since we are interested in the asymptotic behavior of the solution for $t \to \infty$. As in the preceding section, in order to obtain this information it is necessary to transform the given equations to fixed coordinates, and in the resulting equations to look at the asymptotic behavior with respect to t. Under the real conditions of the induction process of zonal noncrucible melting, we generally obtain material of high purity with the help of diffusion and peeling of impurities when the speed of motion of the rod is chosen very small; hence convective heat transfer to the solid phase is found to be not just commensurable with conductive heat transfer, but even to play a lesser role than the latter, and a transition to the one-dimensional problem, similar to that carried out in §4, becomes impracticable. With this limitation, it is essentially useless to employ the approximation methods described in §4. The only practically realizable method for solution of the problem is the method of finite differences. We limit ourselves at present to these remarks, referring to the details in Part 2, Chapter VIII, and in Supplement 3, §6, where we will give illustrative examples of numerical solutions.

CHAPTER II

THERMAL DIFFUSION PROCESSES IN A MEDIUM WITH VARIABLE PHASE STATE

§1. Dissolution of a gas bubble in a liquid [42], [33]

We consider an isothermal process of dissolution of a spherical bubble of gas in a liquid, under the assumption that the redistribution of concentration of the dissolving gas occurs purely by means of diffusion, without convection. In describing the process we will take into account the change of pressure in the bubble resulting from its change of curvature. The magnitude of the surface tension at the gas-liquid interface is regarded as constant during the entire process. We will assume that we have spherical symmetry.

We introduce the following notation:

r is the radial distance from the center of the bubble;

τ is the time;

$R(\tau)$ is the radius of the bubble at time $\tau > 0$; $R_0 = R(0)$;

δ is the solubility coefficient of the gas;

p_0 is the pressure at the liquid-gas interface;

σ is the surface tension;

$\rho = f(p)$ is the density of the gas, where p is the pressure of the gas in the bubble, $p = p_0 + 2\sigma/R(\tau)$;

$c(r, \tau)$ is the density of the dissolved gas in the liquid at the time τ and at radial distance r from the bubble center;

c_s is the saturation density for pressure p in the bubble;

D is the diffusibility of the gas in the liquid.

Moreover,

$$(2.1.1) \quad f(p) = \alpha p; \quad \alpha = \text{const} > 0; \quad c_s = \delta p \equiv \delta \left(p_0 + \frac{2\sigma}{R(\tau)} \right).$$

In this notation the process of dissolution of the bubble is determined by the conditions

$$D\left(\frac{\partial^2 c}{\partial r^2}+\frac{2}{r}\frac{\partial c}{\partial r}\right)=\frac{\partial c}{\partial \tau};\quad R(\tau)<r<\infty;\quad \rho>0;$$

(2.1.2)
$$c|_{r=R(\tau)}=c_s(\tau);\quad c|_{\tau=0}=c_0(r);$$
$$-\frac{\partial}{\partial \tau}\left(\frac{4}{3}\pi R^3(\tau)\rho\right)=-D\cdot 4\pi R^2(\tau)\frac{\partial c}{\partial r}\Bigg|_{r=R(\tau)};\quad R|_{\tau=0}=R_0.$$

The last of conditions (2.1.2) is that of material balance, according to which the mass velocity of the dissolving bubble is equal to the diffusion rate of the gas within the liquid at the surface of the bubble.

To conditions (2.1.2) we add the condition at infinity, namely, that the product $r\cdot c(r,\tau)$ is uniformly bounded, and moreover, that $r\,\partial c/\partial r$ tends uniformly to zero for $r\to\infty$.

We introduce dimensionless variables by letting

$$x=\frac{r}{R_0};\quad t=\frac{D}{R_0^2}\tau;\quad y(t)=\frac{R(\tau)}{R_0};$$

(2.1.3)
$$\frac{u(x,t)}{x}=\frac{c(r,\tau)-c_s(0)}{\delta p_0};\quad \frac{1}{x}u(x,0)=\frac{c_0(r)-c_s(0)}{\delta p_0}=\frac{1}{x}\phi(x);$$
$$\beta=\frac{2\sigma}{R_0 p_0};\quad b=\frac{2}{3}\beta;\quad a=\frac{\delta}{\alpha};\quad c=\beta a.$$

In terms of these variables, problem (2.1.2) assumes the form

(2.1.4$_1$)
$$L_1 u\equiv\left(\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial t}\right)u=0;\quad y(t)<x<\infty;\quad t>0;$$

(2.1.4$_2$)
$$u|_{t=0}=\phi(x);\quad u|_{x=y(t)}=\beta(1-y(t));\quad t>0;$$

(2.1.4$_3$)
$$\dot{y}(t)=\frac{a}{y+b}\frac{\partial u}{\partial x}\Bigg|_{x=y}-\frac{c}{y}\frac{1-y}{y+b};\quad y(0)=1.$$

The condition at infinity gives

(2.1.4$_4$)
$$|u|_{x\to\infty}<\infty;\quad \lim_{x\to\infty}\frac{\partial u}{\partial x}=0.$$

We introduce an approximate solution of the problem, using a modification of the method of L. S. Leĭbenzon [92]. Namely, we determine $v(x,t,\lambda)$ by the conditions

$$L_1 v = 0; \quad y(\lambda) < x < \infty; \quad t > 0; \tag{2.1.5$_1$}$$

$$v|_{t=0} = \phi_1(x); \quad v|_{x=y(\lambda)} = \beta(1 - y(\lambda)); \tag{2.1.5$_2$}$$

$$|v|_{x\to\infty} < \infty; \quad \lim_{x\to\infty} \frac{\partial v}{\partial x} = 0. \tag{2.1.5$_3$}$$

Here

$$\phi_1(x) = \phi(x) \cdot \eta(x - y(\lambda))\eta(x - 1), \tag{2.1.5$_4$}$$

where $\eta(x)$ is the unit Heaviside function.

The boundary $y(\lambda)$ is determined by the conditions

$$\dot{y}(\lambda) = \frac{a}{y(\lambda) + b} \frac{\partial}{\partial x} v(y(\lambda), \lambda, \lambda) - \frac{c(1 - y(\lambda))}{y(\lambda)(y(\lambda) + b)}; \quad y(0) = 1. \tag{2.1.5$_5$}$$

Finally, we make the approximations

$$u(x,t) \cong v(x,t,t); \quad y(t) \cong y(\lambda)|_{\lambda=t}. \tag{2.1.5$_6$}$$

Let

$$g_i(x,\xi,t;\lambda) = E(x - \xi, t) + (-1)^i E(x + \xi - 2y(\lambda), t) \tag{2.1.6}$$

be the Green's function of the ith boundary problem for the heat conduction equation on the halfline $x > y(\lambda)$. Then $v(x,t;\lambda)$ is defined by the equation

$$\begin{aligned} v(x,t;\lambda) = &\int_{\max(1;y(\lambda))}^{\infty} \phi(\xi) g_1(x,\xi,t;\lambda)\, d\xi \\ &+ \beta(1 - y(\lambda)) \int_0^t \frac{\partial}{\partial \xi} g_1(x, y(\lambda), t - \tau; \lambda)\, d\tau. \end{aligned} \tag{2.1.7}$$

Differentiating v with respect to x for $x > y(\lambda)$ and using the conjugacy of g_1 and g_2, we obtain

$$\begin{aligned} \frac{\partial v}{\partial x} = &\phi(z(\lambda)) g_2(x, z(\lambda), t; \lambda) \\ &+ \int_{z(\lambda)}^{\infty} \dot{\phi}(\xi) g_2(x,\xi,t;\lambda)\, d\xi - \beta(1 - y(\lambda)) g_2(x, y(\lambda), t; \lambda). \end{aligned} \tag{2.1.8}$$

Here

$$z(\lambda) = \max(1, y(\lambda)). \tag{2.1.9}$$

Passing to the limit for $x \to y(\lambda) + 0$ in (2.1.8) and introducing the result in (2.1.5$_5$), we obtain an integro-differential equation for the determination of y:

$$\dot{y}(\lambda) = -\frac{c(1-y(\lambda))}{y(\lambda)+b}\left[\frac{1}{y(\lambda)}+\frac{1}{\sqrt{\pi\lambda}}\right]$$

$$+\frac{a}{y(\lambda)+b}\left\{\phi(z(\lambda))g_2(y(\lambda),z(\lambda),\lambda;\lambda)\right. \tag{2.1.10}$$

$$\left.+\int_{z(\lambda)}^{\infty}\dot{\phi}(\xi)g_2(y(\lambda),\xi,\lambda;\lambda)\,d\xi\right\};\quad y(0)=1.$$

We examine the case

$$c_0(r) = c_0 = \text{const.} \tag{2.1.11}$$

Setting

$$\frac{c_0 - c_s(0)}{\delta p_0} = \gamma \tag{2.1.12}$$

we find that

$$\phi(x) = \gamma x;$$

$$\int_{z(\lambda)}^{\infty}\dot{\phi}(\xi)g_2(y(\lambda),\xi,\lambda;\lambda)\,d\xi = \gamma\,\mathrm{erfc}\frac{z(\lambda)-y(\lambda)}{2\sqrt{\lambda}}.$$

Hence

$$\dot{y}(\lambda) = -\frac{c(1-y(\lambda))}{y(\lambda)+b}\left(\frac{1}{y(\lambda)}+\frac{1}{\sqrt{\pi\lambda}}\right)$$

$$+\frac{a}{y(\lambda)+b}\left\{\frac{\gamma z(\lambda)}{\sqrt{\pi\lambda}}\exp\left[-\frac{(y(\lambda)-z(\lambda))^2}{4\lambda}\right]\right. \tag{2.1.13}$$

$$\left.+\gamma\,\mathrm{erfc}\frac{z(\lambda)-y(\lambda)}{2\sqrt{\lambda}}\right\};\quad y(0)=1.$$

The character of the process is essentially dependent on the sign of γ. Assume first that $\gamma > 0$. Then

$$\lim_{\lambda\to 0}(\sqrt{\lambda}\dot{y}(\lambda)) = \frac{a\gamma}{1+b}\lim_{\lambda\to 0}\exp\left[-\frac{(y(\lambda)-z(\lambda))^2}{4\lambda}\right] > 0. \tag{2.1.14}$$

It follows that in a neighborhood of $\lambda = 0$ we have $z(\lambda) = y(\lambda)$, i.e.

$$\dot{y}(\lambda) = \frac{c(y(\lambda)-1)}{y(\lambda)+b}\left(\frac{1}{y(\lambda)}+\frac{1}{\sqrt{\pi\lambda}}\right)+\frac{a\gamma}{y(\lambda)+b}\left(1+\frac{y(\lambda)}{\sqrt{\pi\lambda}}\right)>0. \tag{2.1.15}$$

Hence $y(\lambda)$ is obviously an increasing function of λ for arbitrary $\lambda > 0$. Thus if $\gamma > 0$, then (2.1.15) is fulfilled for arbitrary λ.

Now let us take $\gamma < 0$. In this case,

$$\lim_{\lambda\to 0}\sqrt{\lambda}\dot{y}(\lambda) < 0. \tag{2.1.16}$$

Thus we conclude that for arbitrary $\lambda > 0$

$$\begin{aligned}\dot{y}(\lambda) = &-\frac{c(1-y(\lambda))}{y(\lambda)+b}\left(\frac{1}{y(\lambda)}+\frac{1}{\sqrt{\pi\lambda}}\right)\\ &+\frac{a}{b+y(\lambda)}\left[\frac{\gamma}{\sqrt{\pi\lambda}}\exp\left(-\frac{(1-y(\lambda))^2}{4\lambda}\right)+\gamma\operatorname{erfc}\frac{1-y(\lambda)}{2\sqrt{\lambda}}\right]<0.\end{aligned} \tag{2.1.17}$$

We note also that the inequalities $\gamma \gtrless 0$ are respectively equivalent to

$$c_0 \gtrless \delta\left(p_0+\frac{2\sigma}{R_0}\right). \tag{2.1.18}$$

At the same time, the density of saturation of the dissolved gas is δp_0 for pressure p_0. Consequently the process can be accompanied by a monotonically decreasing bubble, even though in this case the solution is initially oversaturated. Conversely, the solution can be shown to be unstable in the sense that $R(\tau) \to \infty$ for $\tau \to \infty$ despite the fact that initially it is not saturated.

We note that this result can be obtained rigorously, without using Leĭbenzon's quasi-stationary approximation (see [42]). Thus Leĭbenzon's method gives a qualitatively valid result, although quantitatively it contains errors.[1)]

§2. Dynamics of one-dimensional nonisothermal evaporation of an ideal liquid mixture [145]

We consider a liquid mixture of $n+1$ components, consisting of a volatile solvent, denoted by the index 0, and n chemically noninteracting

[1)] For the results of numerical calculations by Leĭbenzon's method, as well as for the solution to the problem in an exact treatment, see Supplement 3, §4.

volatile components. We denote by $\bar{u}$, $\bar{v}$, $\bar{c}_i$ and $\bar{\sigma}_i$ the absolute temperature of the liquid and gas phases, and the molar concentration of the ith component in the liquid and gas phases, respectively. We assume that the gas phase consists of a vapor of all components of the mixture, and negligibly small amounts of gas dissolved in the liquid phase (such as air); we assign the index -1 to the gas phase. We also assume that initially the liquid phase fills the halfline $-\infty < z < 0$, and the gas phase fills the line $0 < z < \infty$. We also assume that in the liquid and gas phases a homogeneous source of heat with density Q_i is present. The indices $i = 1, 2$ are assigned to the liquid and gas phases respectively. In addition, we suppose that at the evaporation surface $z = \zeta(\tau)$ there is a heat source of density $\bar{q}^*$. The magnitude of the latent heat of evaporation is not included in $\bar{q}^*$.[2)]

We suppose that

a) The liquid phase is an ideal mixture obeying the Rayleigh-Henry law.

b) The distributions of temperature and concentration in the liquid and gas phases obey the equations of molecular heat conduction and diffusion, respectively. Convective and radiative heat transfer will be ignored, as will be heat transfer by diffusion, thermal diffusion and the discharge of heat from the mixture due to variable concentration. The parameters characterizing the process are taken as constant throughout.

With these assumptions the distribution of temperature and concentration in the liquid and solid phases will satisfy the following equations:

$$
\begin{aligned}
&\frac{\partial \bar{u}}{\partial \tau} = \bar{b}_1^2 \frac{\partial^2 \bar{u}}{\partial z^2} + \bar{q}_1; \quad -\infty < z < \zeta(\tau); \quad \tau > 0;\\
&\frac{\partial \bar{v}}{\partial \tau} = \bar{b}_2^2 \frac{\partial^2 \bar{v}}{\partial z^2} + \bar{q}_2; \quad \zeta(\tau) < z < \infty; \quad \tau > 0;\\
&\frac{\partial \bar{c}_i}{\partial \tau} = \bar{D}_{i,1} \frac{\partial^2 \bar{c}_i}{\partial z^2}; \quad -\infty < z < \zeta(\tau); \quad \tau > 0; \quad i = 1, 2, \cdots, n;\\
&\frac{\partial \bar{\sigma}_i}{\partial \tau} = \bar{D}_{i,2} \frac{\partial^2 \bar{\sigma}_i}{\partial z^2}; \quad \zeta(\tau) < z < \infty; \quad \tau > 0; \quad i = 0, 1, 2, \cdots, n.
\end{aligned}
\tag{2.2.1}
$$

Here

[2)] If the evaporated gas flows to the open atmosphere and the gas phase can be considered diathermal, then $\bar{q}^*$ will be due to direct solar insolation.

(2.2.1*) $$\bar{q}_i = \frac{Q_i}{c^i \rho^i}; \qquad (i = 1, 2),$$

and b_i^2, $\overline{D}_{i,j}$, c^i and ρ^i are the coefficients of temperature conductivity, diffusion, specific heat and density, respectively.

The conditions on the evaporation front $z = \zeta(\tau)$ are conditions of heat and mass balance, continuity of temperature, and the Rayleigh-Henry law.

Let $\overline{U}_l$ be the unknown temperature at the surface of evaporation. We have

(2.2.2) $$\bar{u}|_{z=\zeta(\tau)} = \bar{v}|_{z=\zeta(\tau)} = \overline{U}_l; \qquad \tau > 0.$$

Let

(2.2.3) $$N_i = \frac{\bar{c}_i}{\sum_{j=1}^{n} \bar{c}_j}; \qquad (i = 0, 1, \cdots, n)$$

be the molar fraction of the ith component of the liquid phase. Under suitable assumptions on the dilution of the solution we can assume that

(2.2.3*) $$N_i = \frac{\bar{c}_i}{\bar{c}_0}, \quad i = 1, 2, \cdots, n; \qquad N_0 = 1 - \sum_{i=1}^{n} N_i.$$

Let $\sigma_{i,m}^l$ and $\sigma_{i,m}^0$ be the concentration of the saturation couple of the ith component in the liquid phase for temperature $\overline{U}_l$ and for the corresponding temperature $\bar{v}^0$. Then by the theorem of Raoult

(2.2.4) $$\bar{\sigma}_i|_{z=\zeta(\tau)} = \bar{\sigma}_{i,m}^l \cdot N_i|_{z=\zeta(\tau)}; \qquad (i = 0, 1, \cdots, n).$$

We note that it is possible to assume

(2.2.5) $$\bar{\sigma}_{i,m}^l = \bar{\sigma}_{i,m}^0 \exp\left\{-\frac{\lambda_i}{R}\left(\frac{1}{\overline{U}_l} - \frac{1}{\bar{v}^0}\right)\right\}.$$

where R is the universal gas constant and λ_i the latent molar heat of vaporization of the ith component.

Let $n_i(\tau)$ be the number of moles of the ith component evaporating from the mixture at the moment τ and let $\bar{k}_i$ $(i = 1, 2)$ be the coefficient of thermal conductivity of the liquid and gas phases, respectively. Then the expressions

(2.2.6) $$q^1 = k_1 \frac{\partial \bar{u}}{\partial z}\bigg|_{z=\zeta(\tau)}; \quad q^2 = -\bar{k}_2 \frac{\partial \bar{v}}{\partial z}\bigg|_{z=\zeta(\tau)}; \quad q^3 = \sum_{i=0}^{n} \lambda_i \dot{n}_i(\tau)$$

define the corresponding amounts of heat entering the interior of the liquid (q^1) and gas (q^2) phases per unit time at the phase boundary, and the amount of latent heat of vaporization of the mixture (q^3). Obviously we have

$$q^1 + q^2 = q^* - q^3,$$

i.e.

(2.2.7) $$\left[\bar{k}_1 \frac{\partial \bar{u}}{\partial z} - k_2 \frac{\partial \bar{v}}{\partial z}\right]\bigg|_{z=\zeta(\tau)} + \sum_{i=0}^{n} \lambda_i \dot{n}_i(\tau) - \bar{q}^* = 0.$$

It is now obvious that

(2.2.8) $$\dot{n}_i(\tau) = -(\bar{c}_i - \bar{\sigma}_i)\big|_{z=\zeta(\tau)} \dot{\zeta}(\tau) \qquad (i = 0, 1, \cdots, n),$$

and at the same time that

(2.2.9) $$\dot{n}_i(\tau) = \left(\overline{D}_{i,1} \frac{\partial \bar{c}_i}{\partial z} - \overline{D}_{i,2} \frac{\partial \bar{\sigma}_i}{\partial z}\right)\bigg|_{z=\zeta(\tau)} \qquad (i = 0, 1, 2, \cdots, n);$$

moreover, it follows that in (2.2.8) and (2.2.9) we may assume

(2.2.10) $$c_0 \equiv \text{const.}$$

We adjoin suitable initial conditions to the derived system:

(2.2.11) $$\zeta = 0; \quad \bar{u} = \bar{u}_0; \quad \bar{v} = \bar{v}_0; \quad \bar{c}_i = \bar{c}_{i,0}; \quad \bar{\sigma}_i = \bar{\sigma}_{i,0} \quad \text{for } \tau = 0;$$

the conditions at infinity are

(2.2.12) $$\lim_{z \to -\infty} \bar{u} = \bar{u}_0; \quad \lim_{z \to -\infty} \bar{v} = \bar{v}_0; \quad \lim_{z \to -\infty} \bar{\sigma}_i = \bar{\sigma}_{i,0}; \quad \lim_{z \to -\infty} \bar{c}_i = \bar{c}_{i,0}.$$

This is the complete system of equations of the problem.

We now linearize condition (2.2.4) by expanding $\bar{\sigma}_{i,m}^{l} N_i$ in a series in the differences $\bar{v} - \bar{v}_0$ and $\bar{c}_j - \bar{c}_{j,0}$ with the help of (2.2.5), and restricting ourselves to linear terms. For these, moreover, we set $\bar{v}^0 = \bar{v}_0$. We have

$$\bar{\sigma}_i\big|_{z=\zeta(\tau)} = [\alpha_{0,i}^* + \alpha_{1,i}^*(\overline{U}_l - \bar{v}_0) + \alpha_{2,i}^*(\bar{c}_i - \bar{c}_{i,0})]_{z=\zeta(\tau)},$$
$$i = 1, 2, \cdots, n,$$

(2.2.13) $$\bar{\sigma}_0\big|_{z=\zeta(\tau)} = \left[\alpha_{0,0}^* + \alpha_{1,0}^*(\overline{U}_l - \bar{v}_0) - \sum_{j=1}^{n} \alpha_{2j}^*(\bar{c}_j - \bar{c}_{j,0})\right]_{z=\zeta(\tau)},$$

where

$$\alpha_{0,i}^* = \frac{\bar{\sigma}_{i,m}^0}{\bar{c}_0} N_{i,0}; \qquad \alpha_{1,i}^* = \frac{\bar{\sigma}_{i,m}^0 \lambda_i}{R\bar{v}_0^2} N_{i,0}; \qquad \alpha_{2,i}^* = \frac{\bar{\sigma}_{i,m}^0}{\bar{c}_0};$$

(2.2.14)

$$N_{i,0} = \frac{\bar{c}_{i,0}}{\bar{c}_0} \quad (i = 1, 2, \cdots, n); \qquad N_{0,0} = 1 - \sum_{j=1}^{n} N_{j,0}.$$

Thus we will investigate the system (2.2.1), (2.2.2), (2.2.7), (2.2.10)—(2.2.14).

Let D, u^* and h be characteristic constants with the dimensions of the coefficients of diffusion, temperature and length, respectively. We will introduce as dimensionless variables the quantities

$$x = \frac{z}{h}; \qquad t = \frac{D\tau}{h^2}; \qquad a_{i,j}^2 = \frac{\bar{D}_{i,j}}{D} \quad (j = 1, 2); \qquad b_i^2 = \frac{\bar{b}_i^2}{D} \quad (i = 1, 2);$$

$$\lambda = \frac{\bar{k}_2}{\bar{k}_1}; \qquad \gamma_{i,j} = \frac{\bar{c}_{i,0}\lambda_i \bar{D}_{i,j}}{u^* \bar{k}_1}; \quad c_i = \frac{\bar{c}_i - \bar{c}_{i,0}}{\bar{c}_{i,0}} \quad (i = 1, 2, \cdots, n);$$

$$\sigma_i = \frac{\bar{\sigma}_i - \bar{\sigma}_{i,0}}{\bar{c}_{i,0}} \quad (i = 0, 1, 2, \cdots, n); \qquad u = \frac{\bar{u} - \bar{u}_0}{u^*}; \qquad v = \frac{\bar{v} - \bar{v}_0}{u^*};$$

(2.2.15)

$$y(t) = \frac{\zeta(\tau)}{h}; \qquad \kappa_i = 1 - \frac{\bar{\sigma}_{i,0}}{\bar{c}_{i,0}} \quad (i = 1, 2, \cdots, n);$$

$$q_i = \frac{h^2}{Du^*} \bar{q}_i \quad (i = 1, 2); \qquad q^* = \frac{\bar{q}^* h}{\bar{k}_1 u^*}; \qquad \mu = \frac{\bar{v}_0 - \bar{u}_0}{u^*};$$

$$\alpha_{0,i} = \frac{\alpha_{0,i}^* - \bar{\sigma}_{i,0}}{\bar{c}_{i,0}} \quad (i = 0, 1, \cdots, n);$$

$$\alpha_{1,i} = \alpha_{1,i}^* c_{i,0}^{-1} u^*, \quad \alpha_{2,i} = \alpha_{2,i}^* \quad (i = 0, 1, 2, \cdots, n).$$

We now find that u, v, c_i, σ_i and y are to be determined from the conditions

$$(2.2.16_1) \qquad \frac{\partial u}{\partial t} = b_1^2 \frac{\partial^2 u}{\partial x^2} + q_1; \qquad -\infty < x < y(t); \quad t > 0;$$

$$(2.2.16_2) \qquad \frac{\partial v}{\partial t} = b_2^2 \frac{\partial^2 v}{\partial x^2} + q^2; \qquad y(t) < x < \infty; \quad t > 0;$$

$$(2.2.16_3) \qquad \frac{\partial c_i}{\partial t} = a_{i,1}^2 \frac{\partial^2 c_i}{\partial x^2}; \qquad -\infty < x < y(t); \quad t > 0; \quad i = 1, 2, \cdots, n;$$

$$(2.2.16_4) \qquad \frac{\partial \sigma_i}{\partial t} = a_{i,2}^2 \frac{\partial^2 \sigma_i}{\partial x^2}; \qquad y(t) < x < \infty; \quad t > 0; \quad i = 0, 1, 2, \cdots, n;$$

$$y = u = v = c_i = \sigma_j = 0 \quad \text{for } t = 0 \quad \text{and} \quad |x| \to \infty; \qquad i = 1, 2, \cdots, n; \qquad j = 0, \cdots, n; \tag{2.2.16$_5$}$$

$$u = v + \mu \quad \text{for } x = y(t); \quad t > 0; \tag{2.2.16$_6$}$$

$$\begin{aligned} &\sigma_i = \alpha_{0,} + \alpha_{1,i} v + \alpha_{2,i} c_i \quad (i = 1, 2, \cdots, n) \\ &\sigma_0 = \alpha_{0,0} + \alpha_{1,0} v - \sum_{j=1}^{n} \alpha_{2,j} c_j \end{aligned} \quad \text{for } x = y(t); \quad t > 0; \tag{2.2.16$_7$}$$

$$\frac{\partial u}{\partial x} - \lambda \frac{\partial v}{\partial x} + \sum_{j=0}^{n} \left(\gamma_{j,1} \frac{\partial c_j}{\partial x} - \gamma_{j,2} \frac{\partial \sigma_j}{\partial x} \right) - q^* = 0 \qquad \text{for } x = y(t); \quad t > 0; \tag{2.2.16$_8$}$$

$$-(c_i - \sigma_i + \kappa_i)\, \dot{y}(t) = a_{i,1}^2 \frac{\partial c_i}{\partial x} - a_{i,2}^2 \frac{\partial \sigma_i}{\partial x} \qquad \text{for } x = y(t); \quad t > 0; \quad i = 0, 1, \cdots, n. \tag{2.2.16$_9$}$$

In (2.2.16$_9$) we assume

$$c_0 \equiv 0. \tag{2.2.16$_9^*$}$$

We will now construct an approximate solution to the problem, using, as in the previous section, a method of quasi-stationary approximation. Moreover, $\bar{\sigma}_{i,0} = 0$, so that

$$\kappa_i = 1 \qquad (i = 1, 2, \cdots, n). \tag{2.2.17}$$

It is then possible to confine our investigation to just one initial state of the process when c_i and σ_i are close to zero, replacing condition (2.2.16$_9$) by the condition

$$-\dot{y}(t) = \left(a_{i,1}^2 \frac{\partial c_i}{\partial x} - a_{i,2}^2 \frac{\partial \sigma_i}{\partial x} \right) \Bigg|_{x = y(t)}; \quad i = 0, 1, 2, \cdots, n; \quad t > 0; \qquad (c_0 \equiv 0); \quad y(0) = 0. \tag{2.2.18}$$

The system of conditions (2.2.18) can now be written in the form

$$a_{i,1}^2 \frac{\partial c_i}{\partial x} - a_{i,2}^2 \frac{\partial \sigma_i}{\partial x} = -a_{0,2}^2 \frac{\partial \sigma_0}{\partial x} \quad \text{for } x = y(t); \qquad i = 1, 2, \cdots, n; \quad t > 0; \tag{2.2.19}$$

$$\dot{y}(t) = a_{0,2}^2 \frac{\partial \sigma_0}{\partial x} \Bigg|_{x = y(t)}; \quad y(0) = 0. \tag{2.2.20}$$

Thus we consider y in (2.2.16$_i$) ($i = 1, \cdots, 8$) and in (2.2.19) as a constant depending on the parameter τ. Then, obtaining the solutions of this system, we find $y(t)$ from (2.2.20) by setting $\tau = t$ in the right-hand side.

We now set

$$u_1 = u - \int_0^t q_1(\tau)\, d\tau; \quad v_1 = v - \int_0^t q_2(\tau)\, d\tau;$$

$$f_{1,0}(t) = \mu + \int_0^t [q_2(\tau) - q_1(\tau)]\, d\tau; \tag{2.2.21}$$

$$f_i(t) = \alpha_{0,i} + \alpha_{1,i} \int_0^t q_2(\tau)\, d\tau.$$

Then by our assumptions the approximate problem (2.2.16$_i$) ($i = 1, \cdots, 8$), (2.2.19) is reduced to the problem

$$\frac{\partial u_1}{\partial t} = b_1^2 \frac{\partial^2 u_1}{\partial x^2}; \quad -\infty < x < y(\tau); \quad t > 0; \tag{2.2.22$_1$}$$

$$\frac{\partial v_1}{\partial t} = b_2^2 \frac{\partial^2 v_1}{\partial x^2}; \quad y(\tau) < x < \infty; \quad t > 0; \tag{2.2.22$_2$}$$

$$\frac{\partial c_i}{\partial t} = a_{i,1}^2 \frac{\partial^2 c_i}{\partial x^2}; \quad -\infty < x < y(\tau); \quad t > 0; \quad i = 1, 2, \cdots, n; \tag{2.2.22$_3$}$$

$$\frac{\partial \sigma_i}{\partial t} = a_{i,2}^2 \frac{\partial^2 \sigma_i}{\partial x^2}; \quad y(\tau) < x < \infty; \quad t > 0; \quad i = 0, 1, 2, \cdots, n; \tag{2.2.22$_4$}$$

$$u_1 = v_1 = c_i = \sigma_j = 0 \quad \text{for } t = 0; \quad i = 1, 2, \cdots, n; \quad j = 0, 1, 2, \cdots, n; \tag{2.2.22$_5$}$$

$$\left.\begin{aligned} &\sigma_i = f_i(t) + \alpha_{1,i} v_1 + \alpha_{2,i} c_i; \quad i = 1, 2, \cdots, n; \\ &\sigma_0 = f_0(t) + \alpha_{1,0} v_1 - \sum_{j=1}^{n} \alpha_{2,j} c_j; \\ &u_1 = v_1 + f_{1,0}(t); \end{aligned}\right\} \text{for } x = y(\tau); \quad t > 0; \tag{2.2.22$_6$}$$

$$\frac{\partial u_1}{\partial x} - \lambda \frac{\partial v_1}{\partial x} + \sum_{j=1}^{n} \gamma_{j,1} \frac{\partial c_j}{\partial x} - \sum_{j=0}^{n} \gamma_{j,2} \frac{\partial \sigma_j}{\partial x} - q^* = 0$$

$$\text{for } x = y(\tau); \quad t > 0; \tag{2.2.22$_7$}$$

$$a_{i,1}^2 \frac{\partial c_i}{\partial x} - a_{i,2}^2 \frac{\partial \sigma_i}{\partial x} = -a_{0,2}^2 \frac{\partial \sigma_0}{\partial x} \quad \text{for } x = y(t); \quad i = 1, 2, \cdots, n.$$

We now set

$$
\begin{aligned}
&\left.\frac{\partial u_1}{\partial x}\right|_{x=y(\tau)} = \psi_{1,0}(t);\\
&\left.\frac{\partial v_1}{\partial x}\right|_{x=y(\tau)} = \psi(t);\\
&\left.\frac{\partial c_i}{\partial x}\right|_{x=y(\tau)} = \psi_{1,i}(t); \qquad i = 1, 2, \cdots, n;\\
&\left.\frac{\partial \sigma_i}{\partial x}\right|_{x=y(\tau)} = \psi_i(t); \qquad i = 0, 1, 2, \cdots, n.
\end{aligned}
\tag{2.2.23}
$$

Suppose, furthermore, that $g(x, \xi, \alpha^2 t)$ is the Green's function for the second boundary problem of the equation $L_\alpha u = 0$ on the rays $x > y$. We then have

$$
\begin{aligned}
u_1(x,t) &= b_1^2 \int_0^t \psi_{1,0}(\lambda) g(x, y, b_1^2(t-\lambda))\, d\lambda;\\
v_1(x,t) &= -b_2^2 \int_0^t \psi(\lambda) g(x, y, b_2^2(t-\lambda))\, d\lambda;\\
c_i(x,t) &= a_{i,1}^2 \int_0^t \psi_{1,i}(\lambda) g(x, y, a_{i,1}^2(t-\lambda))\, d\lambda; \qquad (i = 1, 2, \cdots, n);\\
\sigma_i(x,t) &= -a_{i,2}^2 \int_0^t \psi_i(\lambda) g(x, y, a_{i,1}^2(t-\lambda))\, d\lambda; \qquad (i = 0, 1, 2, \cdots, n).
\end{aligned}
\tag{2.2.24}
$$

Hence

$$
\begin{aligned}
u_1(y,t) &= \frac{b_1}{\sqrt{\pi}} \int_0^t \frac{\psi_{1,0}(\lambda)\, d\lambda}{\sqrt{t-\lambda}}; \qquad v_1(y,t) = -\frac{b_2}{\sqrt{\pi}} \int_0^t \frac{\psi(\lambda)\, d\lambda}{\sqrt{t-\lambda}};\\
c_i(y,t) &= \frac{a_{i,1}}{\sqrt{\pi}} \int_0^t \frac{\psi_{1,i}(\lambda)\, d\lambda}{\sqrt{t-\lambda}}; \qquad \sigma_i(y,t) = -\frac{a_{i,2}}{\sqrt{\pi}} \int_0^t \frac{\psi_i(\lambda)\, d\lambda}{\sqrt{t-\lambda}}.
\end{aligned}
\tag{2.2.25}
$$

If we introduce (2.2.25) into $(2.2.22_6)$—$(2.2.22_7)$, we obtain

$$
f_{1,0}(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{b_1 \psi_{1,0}(\lambda) + b_2 \psi(\lambda)}{\sqrt{t-\lambda}}\, d\lambda;
\tag{2.2.26$_1$}
$$

$$
f_i(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{b_2 \alpha_{1,i} \psi(\lambda) - a_{i,1} \alpha_{2,i} \psi_{1,i}(\lambda) - a_{i,2} \psi_i(\lambda)}{\sqrt{t-\lambda}}\, d\lambda;
\tag{2.2.26$_2$}
$$

$$(2.2.26_3) \qquad f_0(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{b_2\alpha_{1,0}\psi(\lambda) - a_{0,2}\psi_0(\lambda) + \sum_{j=1}^{n} \alpha_{j,1}\alpha_{2,j}\psi_{1,j}(\lambda)}{\sqrt{t-\lambda}}\, d\lambda;$$

$$(2.2.26_4) \qquad \psi_{1,0}(t) - \lambda\psi(t) + \sum_{j=1}^{n} \gamma_{j,1}\psi_{1,j}(t) - \sum_{j=0}^{n} \gamma_{j,2}\psi_j(t) - q^* = 0;$$

$$(2.2.26_5) \qquad a_{i,1}\psi_{1,i}(t) - a_{i,2}^2\psi_i(t) = -\alpha_{0,2}^2\psi_0(t) \qquad (i = 1, 2, \cdots, n).$$

Solving the system $(2.2.26_1)$—$(2.2.26_3)$ for the integrand function, we obtain

$$b_1\psi_{1,0}(t) + b_2\psi(t) = F_{1,0}(t);$$

$$b_2\alpha_{1,i}\psi(t) - a_{i,1}\alpha_{2,i}\psi_{1,i}(t) - a_{i,2}\psi_i(t) = F_i(t) \qquad (i = 1, 2, \cdots, n); \tag{2.2.27}$$

$$b_2\alpha_{1,0}\psi(t) - a_{0,2}\psi_0(t) + \sum_{j=1}^{n} a_{j,1}\alpha_{2,j}\psi_{1,j}(t) = F_0(t).$$

Here

$$F_{1,0}(t) = \frac{1}{\sqrt{\pi t}} f_{1,0}(0) + \frac{1}{\sqrt{\pi}} \int_0^t \frac{\dot{f}_{1,0}(\tau)\, d\tau}{\sqrt{t-\tau}};$$

$$F_i(t) = \frac{1}{\sqrt{\pi t}} f_i(0) + \frac{1}{\sqrt{\pi}} \int_0^t \frac{\dot{f}_i(\tau)\, d\tau}{\sqrt{t-\tau}}; \tag{2.2.28}$$

$$F_0(t) = \frac{1}{\sqrt{\pi t}} f_0(0) + \frac{1}{\sqrt{\pi}} \int_0^t \frac{\dot{f}_0(\tau)\, d\tau}{\sqrt{t-\tau}}.$$

The system $(2.2.26_i)$ $(i = 4, 5)$, (2.2.27) consists of $2n + 3$ equations in $2n + 3$ unknowns. Its coefficients are constant and continuously dependent on the independent parameters of the process. It can be shown that the determinant of this system is not equal to zero, i.e. that it uniquely determines the values $\psi_{1,0}$, $\psi_{1,i}$, ψ_i and ψ, and together with them, the solution of the problem $(2.2.22_i)$ $(i = 1, \cdots, 7)$ for fixed y. When we have its solution, we find y by integration of (2.2.20).

We note that $f_i(0) \neq 0$ $(i = 0, \cdots, n)$, since $\bar{\sigma}_{i,m}^0 \neq 0$. Consequently for $t \to 0$ the values $\psi_{1,0}, \cdots, \psi$ tend to infinity with the order $t^{-1/2}$, and together with $y(t)$ represent in a neighborhood of $t = 0$ a parabola of the form $y(t) = 2\beta\sqrt{t}$.

We note the case when the original problem admits a similarity solution. Let

(2.2.29) $$q_1 = q_2 = q^* = 0.$$

Then the solution can be written in the form

$$u_1 = A_1 + B_1 \operatorname{erf} \frac{x}{2b_1\sqrt{t}}; \quad v_1 = A_2 + B_2 \operatorname{erf} \frac{x}{2b_2\sqrt{t}};$$

(2.2.30) $$c_i = A_{1,i} + B_{1,i} \operatorname{erf} \frac{x}{2a_{i,1}\sqrt{t}}; \quad \sigma_i = A_{2,i} + B_{2,i} \operatorname{erf} \frac{x}{2a_{i,2}\sqrt{t}};$$

$$y = 2\beta\sqrt{t},$$

where, for the determination of the coefficients A_i, B_i, $A_{i,j}$ and $B_{i,j}$ the boundary conditions (2.2.16$_i$) ($i = 5, \cdots, 9$) give a nonlinear system of equations; the investigation of this system is, in the general case, very difficult. Therefore, even in the case when we have a similarity solution to the problem, it is more appropriate to use some approximate method like the one above. We note that in case (2.2.29) $f_{1,0}$ and every f_i is constant. Consequently in (2.2.28) the integral on the right reduces to zero. Therefore every ψ_i, $\psi_{1,i}$ and ψ is a function of the form const $\cdot t^{-1/2}$. But then it follows that the integral (2.2.24) reduces to an expression of the form (2.2.30); hence it is again confirmed that the quasi-stationary Leĭbenzon approximation yields a qualitatively correct conclusion.

We note that the result given by means of the quasi-stationary approximation is less accurate for a volatile solvent. In the case of a mixture with nonvolatile components we get high precision by this method.

In the case when the initial concentrations of the liquid and gas phases are in equilibrium but a heat source ($q_i \neq 0$, $i = 1, 2$, or $q^* \neq 0$) is present, the interphase boundary will be vaporized with finite speed.

At the present time we limit ourselves to the preceding remarks. An illustrative problem involving the calculation of the solution for the evaporation of a mixture, and admitting a similarity solution, is examined by exact and approximate methods in Supplement 3, §5.

§3. Crystallization of a binary alloy [70]

We consider a binary system (A, B) forming a continuous solid alloy. Let c be the concentration density of the more refractory component B, T the temperature and

(2.3.1) $$c = f_l(T), \quad c = f_s(T)$$

the curves of liquification and solidification under isobaric equilibrium, respectively. (See Figure 3.)

Assume that

$$f_l(T_A) = f_s(T_A) < f_l(T) < f_s(T) < f_l(T_B) = f_s(T_B). \tag{2.3.2}$$

Here T_A and T_B are the melting temperatures of A and B.

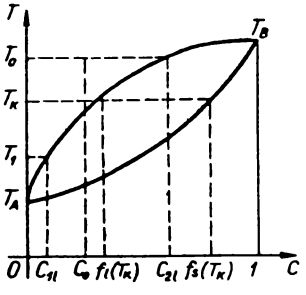

FIGURE 3

The functions $f_l(T)$ and $f_s(T)$ are assumed to be smooth and monotonic increasing. Note that with conditions of equilibrium, in the domain $c < f_l(T)$ the system belongs to the liquid phase, and in the domain $c > f_s(T)$, to the solid phase. In the domain with $f_l(T) < c < f_s(T)$ the system is either heterogeneous or belongs to an unstable supercooled (or supersaturated) state. Let the system (A, B) occupy the line $x > 0$, with an impermeable boundary at $x = 0$ for which

$$\begin{aligned} &T = T_0, \quad c = c_0 \text{ for } t = 0, \quad x > 0; \\ &T = T_1 \qquad\quad \text{for } x = 0, \quad t > 0. \end{aligned} \tag{2.3.3}$$

We suppose that

$$c_{1l} = f_l(T_1) < c_0 < c_{2l} = f_l(T_0), \tag{2.3.4}$$

i.e. that at the initial moment the system belongs to the liquid phase and that at the boundary $x = 0$ the temperature is maintained at a value less than the equilibrium temperature of crystallization of the mixture in its initial form. Hence at the initial moment a crystallization of the mixture is initiated at the boundary $x = 0$.[3)]

The crystallization process of the mixture is studied under the following assumptions:

a) The heat propagation and redistribution of concentration in both phases is due exclusively to molecular heat conduction and diffusion. As above, the thermal diffusion process will be neglected, as will be the heat

[3)] Our whole analysis is carried out in such a way as to be reversible. Thus we assume that the centers of crystallization are formed at the plane $x = 0$ instantaneously and to the necessary amount.

of diffusion due to variations of the concentration.

b) The coefficients of heat, of temperature conduction and of diffusion in each phase are assumed constant. As in the first chapter, we neglect density changes due to phase changes.

We assign the indices s and l to each variable, to denote the solid and liquid phases respectively, and we denote by $x = y(t)$ the interphase boundary. Then the crystallization process of the mixture will be described by the equations

$$(2.3.5_1) \qquad a_s^2 \frac{\partial^2 T_s}{\partial x^2} = \frac{\partial T_s}{\partial t}, \quad D_s \frac{\partial^2 c_s}{\partial x^2} = \frac{\partial c_s}{\partial t}, \quad 0 < x < y(t);$$

$$(2.3.5_2) \qquad a_l^2 \frac{\partial^2 T_l}{\partial x^2} = \frac{\partial T_l}{\partial t}, \quad D_l \frac{\partial^2 c_l}{\partial x^2} = \frac{\partial c_l}{\partial t}, \quad y(t) < x < \infty;$$

$$(2.3.5_3) \qquad T_s|_{x=0} = T_1; \qquad \left.\frac{\partial c_s}{\partial x}\right|_{x=0} = 0;$$

$$(2.3.5_4) \qquad T_l|_{t=0} = T_0; \quad c_l|_{t=0} = c_0; \quad T_l|_{x\to\infty} = T_0; \quad c_l|_{x\to\infty} = c_0;$$

$$(2.3.5_5) \qquad T_s = T_l = T_k; \quad c_s = f_s(T_k); \quad c_l = f_l(T_k) \text{ for } x = y(t);$$

$$(2.3.5_6) \qquad \gamma\rho\dot{y}(t) = k_s \frac{\partial T_s}{\partial x} - k_l \frac{\partial T_l}{\partial x} \text{ for } x = y(t),\ y(0) = 0;$$

$$(2.3.5_7) \qquad [f_s(T_k) - f_l(T_k)]\dot{y}(t) = D_l \frac{\partial c_l}{\partial x} - D_s \frac{\partial c_s}{\partial x} \text{ for } x = y(t).$$

Here T_k denotes the critical temperature of crystallization.

Problem (2.3.5) admits a similarity solution which can be written in the form

$$y(t) = 2\beta\sqrt{t};$$

$$c_s = f_s(T_k); \quad T_s = T_1 + (T_k - T_1)\frac{\operatorname{erf}\dfrac{x}{2a_s\sqrt{t}}}{\operatorname{erf}\left(\dfrac{\beta}{a_s}\right)};$$

$$(2.3.6) \qquad c_l = c_0 + (f_l(T_k) - c_0)\frac{\operatorname{erfc}\dfrac{x}{2\sqrt{D_l t}}}{\operatorname{erfc}\dfrac{\beta}{\sqrt{D_l}}};$$

$$T_l = T_0 + (T_k - T_0)\frac{\operatorname{erfc}\dfrac{x}{2a_l\sqrt{t}}}{\operatorname{erfc}\dfrac{\beta}{a_l}}.$$

This function automatically satisfies every condition of the problem, except conditions $(2.3.5_6)$ and $(2.3.5_7)$. Introducing (2.3.6) into them we obtain a system of two transcendental equations, determining the unknowns β and T_k:

$$\rho\gamma_\beta = \frac{k_s(T_k - T_1)\exp\left(-\frac{\beta^2}{a_s^2}\right)}{a_s\sqrt{\pi}\,\operatorname{erf}\frac{\beta}{a_s}} + \frac{k_l(T_k - T_0)\exp\left(-\frac{\beta^2}{a_l^2}\right)}{a_l\sqrt{\pi}\,\operatorname{erfc}\frac{\beta}{a_l}}, \tag{2.3.7$_1$}$$

$$\beta[f_s(T_k) - f_l(T_k)] = -\frac{\sqrt{D_l}}{\sqrt{\pi}}\frac{[f_l(T_k) - c_0]}{\operatorname{erfc}(\beta\sqrt{D_l})}\exp\left(-\frac{\beta^2}{D_l}\right). \tag{2.3.7$_2$}$$

It is easy to prove this existence of a solution to this system. Actually, solving $(2.3.7_1)$ for T_k, we find that

$$\begin{aligned} T_k = F(\beta) \equiv {} & \frac{\sqrt{\pi}\gamma\rho\beta a_s a_l \exp\left(\frac{\beta^2}{a_l^2}\right)\operatorname{erfc}\frac{\beta}{a_l}\operatorname{erf}\frac{\beta}{a_s}}{k_s a_l \exp\left(\frac{\beta^2}{a_l^2} - \frac{\beta^2}{a_s^2}\right)\operatorname{erfc}\frac{\beta}{a_l} + k_l a_s \operatorname{erf}\frac{\beta}{a_s}} \\ & + \frac{T_1 k_s a_l \exp\left(\frac{\beta^2}{a_l^2} - \frac{\beta^2}{a_s^2}\right)\operatorname{erfc}\frac{\beta}{a_l} + T_0 k_l a_s \operatorname{erf}\frac{\beta}{a_s}}{k_s a_l \exp\left(\frac{\beta^2}{a_l^2} - \frac{\beta^2}{a_s^2}\right)\operatorname{erfc}\frac{\beta}{a_l} + k_l a_s \operatorname{erf}\frac{\beta}{a_s}}. \end{aligned} \tag{2.3.8}$$

It is easily seen that $F(\beta)$ is monotonically increasing,[4] and moreover

[4] Let us prove the monotonicity of $F(\beta)$. We have

$$F(\beta) = T_1 + (T_0 - T_1)\Phi(\beta)(1 + \Phi(\beta))^{-1} + Af_1(\beta/a_l)\Phi(\beta)(1 + \Phi(\beta))^{-1}, \tag{*}$$

where

$$\Phi(\beta) = \frac{k_l a_s \exp(\beta^2/a_s^2)\operatorname{erfc}(\beta/a_s)}{k_s a_l \exp(\beta^2/a_l^2)\operatorname{erfc}(\beta/a_l)}; \quad f_1(\beta) = \beta\exp(\beta^2)\operatorname{erfc}\beta; \quad A = \text{const} > 0.$$

Further,

$$(\exp(x^2)\operatorname{erfc}x)' = 2x\exp(x^2)\operatorname{erfc}x - 2/\sqrt{\pi} < 0.$$

Hence $\Phi(\beta)$ is the ratio of a monotonic increasing to a monotonic decreasing function, i.e. $\Phi(\beta)$ is monotonic increasing. Furthermore,

$$f_1'(x) \equiv f(x) = \exp(x^2)(1 + 2x^2)\operatorname{erfc}x - (2/\sqrt{\pi})x.$$

We have

(2.3.9) $$\lim_{\beta\to 0} F(\beta) = T_1; \qquad \lim_{\beta\to 0} F(\beta) = T_0 + \frac{\rho\gamma a_l^2}{k_l} = T_2.$$

Let

(2.3.10) $$\phi(T) = \frac{f_s(T) - f_l(T)}{c_0 - f_l(T)}$$

and denote by

(2.3.11) $$T = \psi_l(c); \qquad T = \psi_s(c)$$

the equations of the liquid-solid curves for the temperature in terms of the concentration. Then $f_l(\psi_l(c)) \equiv f_s(\psi(c)) \equiv c$. Solving $(2.3.7_2)$ for $\phi(T)$, we find that

(2.3.12) $$\phi(T) = \Phi^*(\beta) \equiv \frac{\sqrt{D_l}}{\beta\sqrt{\pi}} \cdot \frac{\exp(-\beta^2/D_l)}{\operatorname{erfc}(\beta/\sqrt{D_l})}.$$

But $\Phi^*(\beta)$ is monotonically decreasing. Consequently (2.3.12) can be written in the form

(2.3.13) $$\beta = \Psi(\phi(T)),$$

where $\Psi(\phi)$ is monotonically decreasing.

On the other hand, from (2.3.10) it follows that

(2.3.14) $$\lim_{T\to\psi_l(c_0)} \phi = \infty; \qquad \lim_{T\to\psi_s(c_0)} \phi = 1.$$

From (2.3.14) and (2.3.12) it follows, in turn, that

(2.3.15) $$\lim_{T\to\psi_l(c_0)} \beta = 0; \qquad \lim_{T\to\psi_s(c_0)} \beta = \infty.$$

By our assumptions $f_l(T_1) < c_0$. Consequently

(2.3.16) $$T_1 < \psi_l(c_0) < T_0 < T_2.$$

Comparing (2.3.16), (2.3.15) and (2.3.9), we see that there exists a solution to the equations of the system $(2.3.7_1)$, $(2.3.7_2)$, as long as the curves $T = F(\beta)$ and $\beta = \Psi(\phi(T))$ intersect (see Figure 4).

$$\phi = f\exp(-x^2) = (1 + 2x^2)\operatorname{erfc} x - (2/\sqrt{\pi})x\exp(-x^2);$$

$$\lim \phi(x) = 0 \text{ for } x \to \infty; \qquad \phi(0) = 1.$$

But $\phi' = -(4/\sqrt{\pi})\exp(-x^2) + 4x\operatorname{erfc} x$, i.e. $\phi' < 0$. Consequently $\phi > 0$ and $f > 0$. Hence from (*) it is obvious that $F'(\beta) > 0$, which proves the assertion.

For the proof of uniqueness of the solution it would be sufficient to prove monotonicity of the function $\beta = \Psi(\phi(T))$; this, however, is not possible without special limitations on the character of the solid and liquid curves and on the initial concentration.

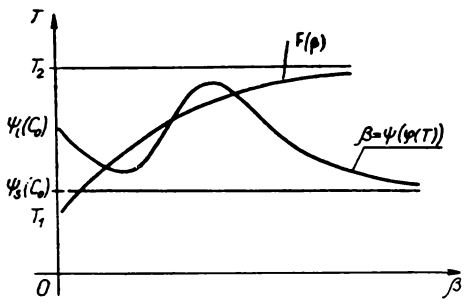

FIGURE 4

We stress that the composition of the solid phase always remains constant. The composition of the liquid phase tends asymptotically to a structure determined figuratively by the points $(T_k, f_l(T_k))$. The temperature of crystallization T_k is automatically less than the temperature $T = \Psi_l(c_0)$ in equilibrium with the initial concentration of the mixture, if the curve $\Psi(\phi(T))$ is monotonic. In this case the crystallization of the binary mixture runs its course under supercooling, a state referred to as "diffusion" by G. P. Ivancov.

Assume now that the process of crystallization of the binary system is capable of resulting in a simple eutectic mixture, preserving all the other assumptions made above.

Let c_E and T_E be the eutectic concentration of component B and the eutectic temperature respectively. Now let

$$c = f_A(T) \text{ for } c < c_E; \quad c = f_B(T) \text{ for } c > c_E; \quad T = \phi(c), \tag{2.3.17}$$

be the equations of the equilibrium phase curves, for the relative volume concentration (in moles per unit) of the component B, and for the temperature, respectively (see Figure 5). For simplicity, we assume that under the formation of the melt there is additional volume. For definiteness we assume that

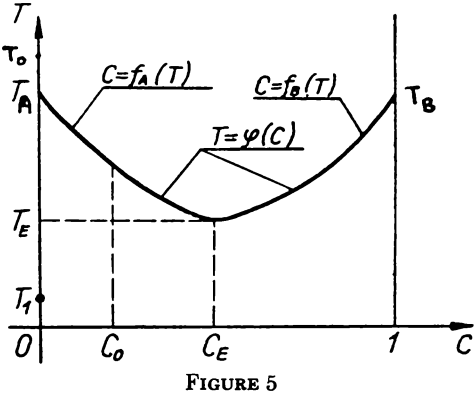

FIGURE 5

$$T_0 > T_A = \phi(0); \quad c_0 < c_E; \quad T_1 < T_E. \tag{2.3.18}$$

In this case the crystallization process is determined by the conditions

$$(2.3.19_1) \qquad a_s^2 \frac{\partial^2 T_s}{\partial x^2} = \frac{\partial T_s}{\partial t},\ 0 < x < y(t); \quad a_l^2 \frac{\partial^2 T_l}{\partial x^2} = \frac{\partial T_l}{\partial t},\ y(t) < x < \infty,$$

$$(2.3.19_2) \qquad D_l \frac{\partial^2 c_l}{\partial x^2} = \frac{\partial c_l}{\partial t},\ y(t) < x < \infty,$$

$$(2.3.19_3) \quad T_s|_{x=0} = T_1; \quad T_l|_{x\to\infty} = T_0; \quad T_l|_{t=0} = T_0; \quad c_l|_{x\to\infty} = c_0; \quad c_l|_{t=0} =$$

$$(2.3.19_4) \qquad T_s|_{x=y(t)} = T_l|_{x=y(t)} = T_k; \quad T_k = \phi(c_k),$$

$$(2.3.19_5) \qquad \gamma\rho \frac{dy}{dt} = \left(k_s \frac{\partial T_s}{\partial x} - k_l \frac{\partial T_l}{\partial x}\right)\Bigg|_{x=y(t)},\ y(0) = 0,$$

$$(2.3.19_6) \qquad c_l|_{x=y(t)} \frac{dy}{dt} = -D_l \frac{\partial c_l}{\partial x}\Bigg|_{x=y(t)},\ t > 0.$$

The last equation obviously expresses the material balance at the boundary $x = y(t)$ of crystallization of the melt, under the assumption that the generated solid phase consists of the component A in pure form, i.e. that diffusion of component B in the direction of the boundary of the melt is within the liquid phase. The concentration of component B in the melt at the phase boundary is again unknown and subject to determination.

We will again write the solution of the problem in similarity form, i.e. we assume

$$(2.3.20) \qquad \begin{aligned} y &= 2\beta\sqrt{t}\,; \\ T_s &= T_1 + (T_k - T_1)\frac{\operatorname{erf}(x/2a_s\sqrt{t})}{\operatorname{erf}(\beta/a_s)}; \\ T_l &= T_0 + (T_k - T_0)\frac{\operatorname{erfc}(x/2a_l\sqrt{t})}{\operatorname{erfc}(\beta/a_l)}; \\ c_l &= c_0 + (c_k - c_0)\frac{\operatorname{erfc}(x/2\sqrt{D_l t})}{\operatorname{erfc}(\beta/\sqrt{D_l})}; \\ c_k &= f_A(T_k). \end{aligned}$$

Introducing (2.3.20) into $(2.3.19_5)$ and $(2.3.19_6)$, we obtain, as earlier,

$$(2.3.21) \qquad \begin{aligned} T_k &= F(\beta_k); \\ c_k &= c_0[1 - \sqrt{\pi}x \exp x^2 \operatorname{erfc} x]^{-1} \equiv \Phi(\beta), \end{aligned}$$

where

$$x = \beta/\sqrt{D_l}$$

and $F(\beta)$ is determined by (2.3.8).

We will let

$$\lim_{\beta\to 0} F(\beta) = T_1; \qquad \lim_{\beta\to\infty} F(\beta) = T_2 > T_0; \tag{2.3.22}$$
$$\lim_{\beta\to 0} \Phi(\beta) = c_0; \qquad \lim_{\beta\to\infty} \Phi(\beta) = \infty.$$

Assume that $F_A^*(T)$ is a monotonically decreasing function, defined on the interval $T_1 \leqq T \leqq T_2$, coincident with $f_A(T)$ for $T_E \leqq T \leqq T_A = \phi(0)$. Since by assumption $T_1 < T_E$, it is possible to define $f_A^*(T)$ such that

$$\lim_{T\to T_1} f_A^*(T) = \infty. \tag{2.3.23}$$

In exactly the same way it can be assumed that

$$f_A^*(T_2) = -1. \tag{2.3.24}$$

We know that

$$c_k = f_A(T_k). \tag{2.3.25}$$

Using the equation

$$f_A^*(T) = c = \Phi(\beta) \tag{2.3.26}$$

we determine T as a monotonic decreasing function of β:

$$T = F^*(\beta) \tag{2.3.27}$$

such that

$$F^*(0) = \phi(c_0); \qquad \lim_{\beta\to\infty} F^*(\beta) = T_1. \tag{2.3.28}$$

But this shows that the system

$$T = F(\beta); \qquad T = F^*(\beta) \tag{2.3.29}$$

has a uniquely determined solution (see Figure 6). Let $\beta = \beta_k$ be a root of the system (2.3.29). We have two cases:

$$\text{a) } c_0 < \Phi(\beta_k) < c_E; \qquad \text{b) } c_E < \Phi(\beta_k) < \infty. \tag{2.3.30}$$

In the first case

$$c_k = \Phi(\beta_k) = f_A(T_k); \qquad T_k = F(\beta_k) \tag{2.3.31}$$

is the desired concentration and temperature of crystallization. Hence the similarity solution to the problem exists and is unique. In the second case it is found that the conditions

$$c_k = f_A(T_k); \qquad c_k < c_E \tag{2.3.32}$$

are incompatible with the remaining conditions of the problem. This shows that there does not exist a similarity solution to the problem of crystallization of a binary system with simple eutectic in which the solid phase consists purely of component A. In exactly the same way it is impossible to have crystallization with the solid phase composed solely of component B, since by our assumptions its initial concentration is below eutectic, and the transition of component B from liquid to solid phase brings about a lack of this component in the melt. Thus, in the case considered, crystallization of the melt cannot possibly begin at the moment $t = 0$. In the course of some time the melt will cool without formation of the solid phase, after which crystallization will begin spontaneously from within the material, due to supercooling of the melt. We limit ourselves to these remarks, since the investigation of similar irreversible processes is beyond the framework of problems of Stefan type. We merely observe that one of the problems of the theory is the determination of conditions under which supercooling is unavoidable.

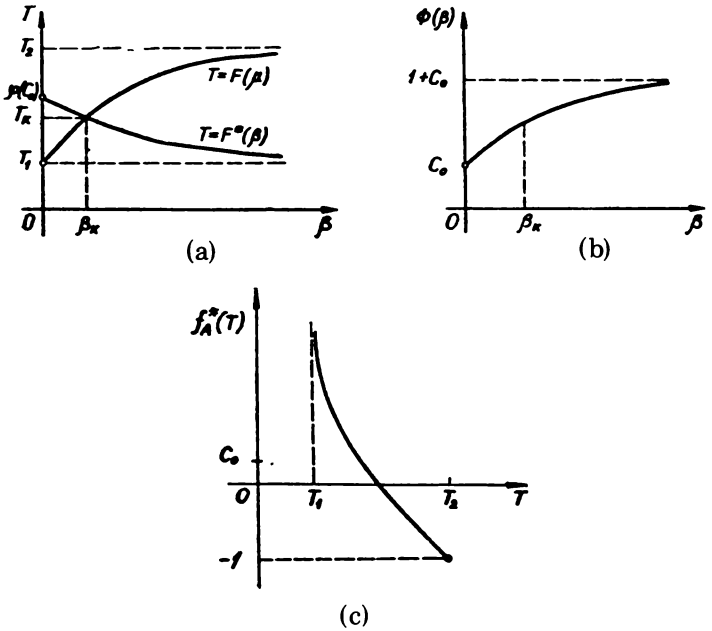

FIGURE 6

CHAPTER III

PROBLEMS OF THE THEORY OF FILTRATION

§1. Forcing of a hydraulic solution into the ground. (Problem of Verigin [191])

Given that an elastic hydraulic solution is being forced into water-saturated ground of porosity m, let it be required to determine the position of the boundary of penetration of the solution into the ground and the distribution of pressure in the regions occupied by the injected solution and by the displaced ground-water. We assume that the speed of filtration obeys Darcy's law, and that the pressure in each of the two phases obeys the equation of piezoconductivity.

We assume that the injection of the solution takes place through a tunnel, so that the filtration process can be assumed one-dimensional, and on the front face of the injecting tunnel we assume that the pressure is given as a constant $p = P_1$. Let the x-axis be in a direction perpendicular to the axis of the tunnel and let $x = y(t)$ be the equation of the boundary of separation of the domain occupied by the injected solution and the displaced ground-water. We assign the index $i = 1$ to the domain $0 < x < y(t)$ and the index $i = 2$ to the domain $y(t) < x < \infty$. Then the following equations must be satisfied:

$$a_1^2 \frac{\partial^2 p_1}{\partial x^2} = \frac{\partial p_1}{\partial t}; \quad 0 < x < y(t); \quad t > 0; \tag{3.1.1$_1$}$$

$$a_2^2 \frac{\partial^2 p_2}{\partial x^2} = \frac{\partial p_2}{\partial t}; \quad y(t) < x < \infty; \quad t > 0; \tag{3.1.1$_2$}$$

$$p_1|_{x=0} = P_1; \qquad p_2|_{t=0} = P_0 < P_1. \tag{3.1.1$_3$}$$

On the boundary $x = y(t)$ we clearly have the conditions of continuity of pressure and flux.

Let u_1 and u_2 be the velocity of filtration of the injected solution and the ground-water respectively. By assumption, for each of the liquids the Darcy law holds with constant filtration coefficients k_1 and k_2:

$$u_i = -k_i \frac{\partial p_i}{\partial x} \quad (i = 1, 2). \tag{3.1.2}$$

Thus we have

$$p_1 = p_2; \qquad k_1 \frac{\partial p_1}{\partial x} = k_2 \frac{\partial p_2}{\partial x} \text{ for } x = y(t). \tag{3.1.3}$$

It is obvious, finally, that if the residual water saturation is equal to zero, then on the front $x = y(t)$ we have the condition of material balance

$$m\dot{y}(t) = -k_i \frac{\partial p_i}{\partial x}\bigg|_{x=y(t)} \quad (i = 1, 2). \tag{3.1.4}$$

Finally, we take

$$y(0) = 0. \tag{3.1.5}$$

Problem $(3.1.1_i)$ $(i = 1, 2, 3)$, (3.1.3)—(3.1.5) admits a similarity solution. We will seek this solution in the form

$$\begin{aligned} y &= 2\beta\sqrt{t}; \\ p_1 &= P_1 + A \operatorname{erf}(x/2a_1\sqrt{t}); \\ p_2 &= P_0 + B \operatorname{erfc}(x/2a_2\sqrt{t}). \end{aligned} \tag{3.1.6}$$

For these functions conditions $(3.1.1_i)$ $(i = 1, 2, 3)$ are automatically satisfied. Condition (3.1.3) gives

$$P_1 + A \operatorname{erf}\frac{\beta}{a_1} = P_0 + B \operatorname{erfc}\frac{\beta}{a_2};$$

$$k_1 \frac{A}{a_1} \exp\left(-\frac{\beta^2}{a_1^2}\right) = -k_2 \frac{B}{a_2} \exp\left(-\frac{\beta^2}{a_2^2}\right),$$

whence

$$\begin{aligned} A = -(P_1 - P_0)\lambda \exp\left(-\frac{\beta^2}{a_2^2}\right) \Bigg[\exp\left(-\frac{\beta^2}{a_1^2}\right) \operatorname{erfc}\frac{\beta}{a_2} \\ + \lambda \exp\left(-\frac{\beta^2}{a_2^2}\right) \operatorname{erf}\frac{\beta}{a_1} \Bigg]^{-1}; \\ B = (P_1 - P_0) \exp\left(-\frac{\beta^2}{a_1^2}\right) \Bigg[\exp\left(-\frac{\beta^2}{a_1^2}\right) \operatorname{erfc}\frac{\beta}{a_2} \\ + \lambda \exp\left(-\frac{\beta^2}{a_2^2}\right) \operatorname{erf}\frac{\beta}{a_1} \Bigg]^{-1}. \end{aligned} \tag{3.1.7}$$

Here

$$\lambda = k_2 a_1 / k_1 a_2. \tag{3.1.7*}$$

Substituting (3.1.7) into (3.1.6) and (3.1.4), we obtain

$$m\beta = \frac{k_2}{a_2\sqrt{\pi}} \cdot \frac{(P_1 - P_0)\exp\left[-\left(\frac{\beta^2}{a_1^2} + \frac{\beta^2}{a_2^2}\right)\right]}{\exp\left(-\frac{\beta^2}{a_1^2}\right)\operatorname{erfc}\frac{\beta}{a^2} + \lambda\exp\left(-\frac{\beta^2}{a_2^2}\right)\operatorname{erf}\frac{\beta}{a_1}} \equiv f(\beta). \tag{3.1.8}$$

We have

$$f(0) = \frac{k^2}{a_2\sqrt{\pi}}(P_1 - P_0) > 0; \quad \lim_{\beta\to\infty} f(\beta) = 0; \quad f'(\beta) < 0. \tag{3.1.9}$$

Consequently equation (3.1.8) and hence our problem has a unique solution.

We note that this problem of Verigin has the same significance for the theory of filtration as the example of Stefan [167] for the theory of heat conduction. The two problems are related to each other in the same way as ordinary single-layer problems are related to ordinary two-layer problems under conjugacy conditions of the type of (3.1.3). But problems of the latter type turn out to be essentially more complicated, and this is especially true for the Verigin problem in its general form, in which conjugacy conditions must be satisfied on a boundary unknown in advance and the conditions are lacking to guarantee the existence of a similarity solution. A more complicated problem of this kind is discussed in the following section.

§2. Advance of a water-oil interface in an elastic regime [138]

We now investigate the filtration of water and oil in an oil-bearing geological stratum in an elastic regime. In the displacement of oil by water we must distinguish three regions: the region situated above the initial oil-bearing contour and characterized by one-hundred-percent saturation of water; the region inside the current oil-bearing contour, with oil concentration equal to the initial oil concentration of the oil-bearing part of the stratum, and the area between these regions, in which the actual process of dilution of oil by water runs its course. The oil content in this intermediate zone changes, generally speaking, with time and tends to reach a limiting state, called the residual oil saturation. The interval of time from the moment of arrival of the water-oil contact at a given point of this intermediate zone until the end of the process of dilution of the oil at this point (i.e. until the attainment of the limiting value of oil saturation) depends on the properties of the rock, the water and the oil, and in general this dependence is directly related to the

presence of various materials lowering the surface tension at the water-oil boundary. In individual cases the process is limited, in practice, to the initial stage of piston-like action by the water. This stage is characterized by the fact that if the time is counted from the moment of passage of the oil-water contact boundary past a fixed element of volume, the degree of oil saturation of this volume does not depend on time. This means that the flow in the intermediate filtration zone is of one phase. Through this zone there filters only water, while the oil in it is stationary. Nevertheless it is necessary to distinguish the water-filled part and the intermediate part of the stratum, the reason being that, since the residual oil-concentration in the intermediate zone is greater than zero, the rate of penetration by water in that zone is lower than in the water-filled zone.

Thus we consider a one-dimensional process of filtration in a homogeneous stratum in an elastic regime. The sampling of oil is assumed to be produced at the point $x_1 = 0$. The stratum under consideration is chosen as the halfspace $x_1 < 0$. Let $x_1 = y_1(t_1)$ be the position of the oil-water contact at the moment $t_1 > 0$ and let $y_1(0) = -l$ be its initial position. To the water region we attach the index $i = 0$ $(-\infty < x_1 < -l)$, to the intermediate region $i = 1$ $(-l < x_1 < y_1(t_1))$, and to the oil region $i = 2$ $(y_1(t_1) < x_1 < 0)$. Under certain appropriate assumptions we have, for $t_1 > 0$,

$$
\begin{aligned}
&u_0 = -\frac{k_0}{\mu_0}\frac{\partial p_0}{\partial x_1}; \quad a_0^2\frac{\partial^2 p_0}{\partial x_1^2} = \frac{\partial p_0}{\partial t_1}; \quad -\infty < x_1 < -l;\\
&u_1 = -\frac{k_0}{\mu_1}\frac{\partial p_1}{\partial x_1}; \quad a_1^2\frac{\partial^2 p_1}{\partial x_1^2} = \frac{\partial p_1}{\partial t_1}; \quad -l < x_1 < y_1(t_1);\\
&u_2 = -\frac{k_2}{\mu_2}\frac{\partial p_2}{\partial x_1}; \quad a_2^2\frac{\partial^2 p_2}{\partial x_1^2} = \frac{\partial p_2}{\partial t_1}; \quad y_1(t_1) < x_1 < 0.
\end{aligned}
\tag{3.2.1}
$$

Here k_i, a_i^2 and μ_i denote the coefficients of permeability, piezoconductivity and viscosity of the filtering liquid in the ith zone. Thus $\mu_0 = \mu_1$ denotes the viscosity of water and μ_2 the viscosity of oil; u_i and p_i denote the filtration velocity and pressure in the ith zone.

On the boundaries $x_1 = -l$ and $x_1 = y_1(t_1)$ we have conditions of continuity of flux and pressure

$$
\begin{aligned}
&u_0 = u_1, \quad p_0 = p_1 \text{ for } x_1 = -l;\\
&u_1 = u_2, \quad p_1 = p_2 \text{ for } x_1 = y_1(t_1).
\end{aligned}
\tag{3.2.2}
$$

Finally, the velocity of displacement of the water-oil contact is determined

from the condition of material balance

$$m(1-\alpha-\beta)\dot{y}_1(t_1) = u_1 \text{ for } x_1 = y_1(t_1). \tag{3.2.3}$$

Here $\alpha = \text{const.}$ denotes the content of the underground water, $\beta = \text{const.}$ the residual oil saturation in the transition zone, and m the porosity of the stratum.

At the initial moment $t_1 = 0$ the pressure in the entire stratum is assumed constant. Similarly, we assume that the pressure is a given constant value at the sampling point $x_1 = 0$. The zero reference point for calculation of pressure is so chosen that $p_2(0, t_1) = 0$.

Thus we will solve problem (3.2.1)—(3.2.3) for initial and boundary conditions

$$p_i|_{t=0} = P = \text{const} \ (i = 0,1,2); \quad p_2|_{x_1=0} = 0; \quad y_1(0) = -l. \tag{3.2.4}$$

We introduce the dimensionless variables

$$x = \frac{x_1}{l}; \quad t = \frac{a_0^2}{l^2}t_1; \quad y(t) = \frac{y_1(t_1)}{l};$$

$$v_i = \frac{p_i}{P}; \quad \alpha_i^2 = \frac{a_i^2}{a_0^2}; \quad \lambda_i = \frac{k_i}{\mu_i}; \quad \gamma = \frac{\lambda_1 P}{a_0^2 m(1-\alpha-\beta)}; \quad i = 0,1,2. \tag{3.2.5}$$

Substituting (3.2.5) into (3.2.i) ($i = 1,2,3,4$) and eliminating u_i, we obtain

$$\alpha_0^2 \frac{\partial^2 v_0}{\partial x^2} = \frac{\partial v_0}{\partial t}; \quad -\infty < x < -1; \quad t > 0;$$

$$\alpha_1^2 \frac{\partial^2 v_1}{\partial x^2} = \frac{\partial v_1}{\partial t}; \quad -1 < x < y(t); \quad t > 0; \tag{3.2.6}$$

$$\alpha_2^2 \frac{\partial^2 v_2}{\partial x^2} = \frac{\partial v_2}{\partial t}; \quad y(t) < x < 0; \quad t > 0.$$

$$v_0 = v_2 = 1 \text{ for } t = 0; \quad v_2 = 0 \text{ for } x = 0;$$

$$v_0 = v_1; \quad \lambda_0 \frac{\partial v_0}{\partial x} = \lambda_1 \frac{\partial v_1}{\partial x} \text{ for } x = -1; \quad t > 0; \tag{3.2.6*}$$

$$v_1 = v_2; \quad \lambda_1 \frac{\partial v_1}{\partial x} = \lambda_2 \frac{\partial v_2}{\partial x} \text{ for } x = y(t); \quad t > 0;$$

$$\dot{y}(t) = -\gamma \frac{\partial v_1}{\partial x} \text{ for } x = y(t); \quad y(0) = -1. \tag{3.2.7}$$

We shall give the approximate solution to problem (3.2.6) by restricting ourselves to an examination of the initial stage of the process and using a quasi-stationary approximation as above in Chapter II, §1. Namely, we consider the problem (3.2.6), assuming in it that $y(\tau)$ is a constant in the interval $(-l, 0)$ and considering τ as a parameter. We therefore write

$$v_i = v_i(x, t; \tau) \tag{3.2.8}$$

in place of $v_i(x, t)$. Given $v_i(x, t; \tau)$, we determine $y(t)$ from the condition

$$\dot{y}(t) = -\gamma \frac{\partial}{\partial x} v_1(x, t; t)\big|_{x=y(t)}; \quad y(0) = -1. \tag{3.2.9}$$

Since the pressure at the initial moment is continuous, we shall assume

$$v_1(x, 0; \tau) = 1. \tag{3.2.10}$$

Let w_i be the Laplace-Carson transform of v_i:

$$w_i(x, p) = p \int_0^\infty \exp(-pt) v_i(x, t; \tau)\, dt. \tag{3.2.11}$$

Then (3.2.6) will be written in terms of the transforms in the form

$$\begin{aligned} &\alpha_0^2 \ddot{w}_0 - p w_0 + p = 0; \quad -\infty < x < -1; \\ &\alpha_1^2 \ddot{w}_1 - p w_1 + p = 0; \quad -1 < x < y = y(\tau); \\ &\alpha_2^2 \ddot{w}_2 - p w_2 + p = 0; \quad y < x < 0. \end{aligned} \tag{3.2.12}$$

From the requirement of boundedness of w_i at infinity we find that

$$\begin{aligned} w_0 &= 1 + A \exp\left(\frac{x\sqrt{p}}{\alpha_0}\right); \\ w_1 &= 1 + B \exp\left(\frac{x\sqrt{p}}{\alpha_1}\right) + C \exp\left(-\frac{x\sqrt{p}}{\alpha_1}\right); \\ w_2 &= 1 - \operatorname{ch}\frac{x\sqrt{p}}{\alpha_2} + D \operatorname{sh}\frac{x\sqrt{p}}{\alpha_2}. \end{aligned} \tag{3.2.13}$$

Introducing (3.2.13) into the conditions at the boundaries $x = -1$ and $x = y$, we obtain the system

$$A\exp\left(-\frac{\sqrt{p}}{\alpha_0}\right) = B\exp\left(-\frac{\sqrt{p}}{\alpha_1}\right) + C\exp\left(\frac{\sqrt{p}}{\alpha_0}\right);$$

$$\frac{\lambda_0}{\alpha_0}A\exp\left(-\frac{\sqrt{p}}{\alpha_0}\right) = \frac{\lambda_1}{\alpha_1}\left[B\exp\left(-\frac{\sqrt{p}}{\alpha_1}\right) - C\exp\left(\frac{\sqrt{p}}{\alpha_0}\right)\right];$$

$$B\exp\left(\frac{y\sqrt{p}}{\alpha_1}\right) + C\exp\left(-\frac{y\sqrt{p}}{\alpha_1}\right)$$

(3.2.14)

$$= D\,\mathrm{sh}\left(\frac{y\sqrt{p}}{\alpha_2}\right) - \mathrm{ch}\left(\frac{y\sqrt{p}}{\alpha_2}\right);$$

$$\frac{\lambda_1}{\alpha_1}\left[B\exp\left(\frac{y\sqrt{p}}{\alpha_1}\right) - C\exp\left(-\frac{y\sqrt{p}}{\alpha_1}\right)\right]$$

$$= \frac{\lambda_2}{\alpha_2}\left(D\,\mathrm{ch}\frac{y\sqrt{p}}{\alpha_2} - \mathrm{sh}\frac{y\sqrt{p}}{\alpha_2}\right).$$

We find

(3.2.15) $$\mu_i = \frac{\lambda_i\alpha_{i-1}}{\lambda_{i+1}\alpha_i} \quad (i = 1,2); \qquad \delta_i = \frac{1-\mu_i}{1+\mu_i} \quad (i = 1,2).$$

Hence

$$A = B(1-\delta_1)\exp\left[\sqrt{p}\left(\frac{1}{\alpha_0} - \frac{1}{\alpha_1}\right)\right];$$

$$B = \mu_2\left[\exp\left(\frac{y\sqrt{p}}{\alpha_1}\right)\psi_1 + \delta_1\exp\left(-\frac{y+2}{\alpha_1}\sqrt{p}\right)\psi_2\right]^{-1};$$

(3.2.16) $$C = -\delta_1 B\exp\left(-\frac{2\sqrt{p}}{\alpha_1}\right);$$

$$D = \frac{\exp\left(\frac{y\sqrt{p}}{\alpha_1}\right)\psi_3 + \delta_1\exp\left(-\frac{y+2}{\alpha_1}\sqrt{p}\right)\psi_4}{\exp\left(\frac{y\sqrt{p}}{\alpha_1}\right)\psi_1 + \delta_1\exp\left(-\frac{y+2}{\alpha_1}\sqrt{p}\right)\psi_2},$$

where

(3.2.16*)

$$\psi_1 = \mathrm{sh}\frac{y\sqrt{p}}{\alpha_2} - \mu_2\mathrm{ch}\frac{y\sqrt{p}}{\alpha_2}; \quad \psi_2 = \mu_2\mathrm{ch}\frac{y\sqrt{p}}{\alpha_2} + \mathrm{sh}\frac{y\sqrt{p}}{\alpha_2};$$

$$\psi_3 = \mathrm{ch}\frac{y\sqrt{p}}{\alpha_2} - \mu_2\mathrm{sh}\frac{y\sqrt{p}}{\alpha_2}; \quad \psi_4 = \mathrm{ch}\frac{y\sqrt{p}}{\alpha_2} + \mu_2\mathrm{sh}\frac{y\sqrt{p}}{\alpha_2}.$$

Since we are investigating the process only for small $t > 0$, it is sufficient to compute the values of the coefficients A, B, C and D only for large $\mathrm{Re}\, p$. This means that for y close to -1 we may assume

$$\text{(3.2.17)} \qquad \mathrm{th}\frac{y\sqrt{p}}{\alpha_2} \cong -1; \quad \mathrm{ch}\frac{y\sqrt{p}}{\alpha_2} \cong -\mathrm{sh}\frac{y\sqrt{p}}{\alpha_2} = \frac{1}{2}\exp\left(-\frac{y\sqrt{p}}{\alpha_2}\right).$$

Thus we find that approximately

$$\text{(3.2.18)} \qquad D \cong -1; \quad B \cong -(1-\delta_2)\frac{\exp\left[y\sqrt{p}\left(\frac{1}{\alpha_2}-\frac{1}{\alpha_1}\right)\right]}{1+\delta_1\delta_2\exp\left[-2(y+1)\frac{\sqrt{p}}{\alpha_1}\right]},$$

whence

$$\text{(3.2.19}_0\text{)} \qquad w_0(x,p;\tau) = 1-(1-\delta_1)(1-\delta_2)\frac{\exp\left[\sqrt{p}\left(\frac{x+1}{\alpha_0}+\frac{y}{\alpha_2}-\frac{y+1}{\alpha_1}\right)\right]}{1+\delta_1\delta_2\exp\left[-2(y+1)\frac{\sqrt{p}}{\alpha_1}\right]};$$

$$\text{(3.2.19}_1\text{)} \qquad w_1(x,p;\tau) = 1-(1-\delta_2)\cdot\frac{\exp\left[\left(\frac{x-y}{\alpha_1}+\frac{y}{\alpha_2}\right)\sqrt{p}\right]-\delta_1\exp\left[\left(\frac{y}{\alpha_2}-\frac{2+x+y}{\alpha_1}\right)\sqrt{p}\right]}{1+\delta_1\delta_2\exp\left[-2(y+1)\frac{\sqrt{p}}{\alpha_1}\right]};$$

$$\text{(3.2.19}_2\text{)} \qquad w_2(x,p;\tau) = 1-\exp\left(\frac{x\sqrt{p}}{\alpha_1}\right); \quad y = y(\tau).$$

If we expand $[1+\delta_1\delta_2\exp(-2(y+1)\sqrt{p}/\alpha_1)]^{-1}$ in a series, using the relation

$$\exp(-|a|\sqrt{p}) \div \mathrm{erfc}\frac{|a|}{2\sqrt{t}}$$

between the Laplace-Carson transform and the original function [**50**] and taking into account the relation $\alpha_0 = 1$ obtained from (3.2.5), we find that

$$v_0(x,t;\tau) \cong 1-(1-\delta_1)(1-\delta_2)\sum_{n=0}^{\infty}(-1)^n\delta_1^n\delta_2^n \cdot \operatorname{erfc}\left[\frac{1}{2\sqrt{t}}\left(\frac{2n+1}{\alpha_1}(1+y)-\frac{y}{\alpha_2}-1-x\right)\right];$$

$$v_1(x,t;\tau) \cong 1-(1-\delta_2)\sum_{n=0}^{\infty}(-1)^n\delta_1^n\delta_2^n \cdot \left\{\operatorname{erfc}\left[\frac{1}{2\sqrt{t}}\left(\frac{y+x+2n(y+1)}{\alpha_1}-\frac{y}{\alpha_2}\right)\right] - \delta_1\operatorname{erfc}\left[\frac{1}{2\sqrt{t}}\left(\frac{x+1+(2n+1)(y+1)}{\alpha_1}-\frac{y}{\alpha_2}\right)\right]\right\}; \tag{3.2.20}$$

$$v_2(x,t;\tau) \cong 1-\operatorname{erfc}\frac{|x|}{2\alpha_2\sqrt{t}}; \quad y=y(\tau).$$

Differentiating v_1 with respect to x, setting $x=y(\tau)$, $\tau=t$ and introducing the result in (3.2.9), we obtain

$$\dot{y}(t)=\frac{\gamma(1-\delta_2)}{\alpha_1\sqrt{\pi t}}\sum_{n=0}^{\infty}(-1)^n\delta_1^n\delta_2^n \cdot \left\{\exp\left[-\frac{1}{4t}\left(\frac{2n}{\alpha_1}(y(t)+1)-\frac{y(t)}{\alpha_2}\right)^2\right] + \delta_1\exp\left[-\frac{1}{4t}\left(\frac{2n+2}{\alpha_1}(y(t)+1)-\frac{y(t)}{\alpha_2}\right)^2\right]\right\}; \tag{3.2.21}$$

$$y(0)=-1,$$

whence the approximate value $y(t)$ is found by numerical integration.

Another approximation is found by neglecting the compressibility of the liquids in the stratum and the deformation of the stratum in the intermediate and oil zones. We note that such an approximation is of interest because of the fact that in many systems of this kind the elasticity of the oil-producing part is generally much smaller than the elasticity of the system as a whole. In this case it is entirely allowable to consider the oil-bearing part of the stratum and the liquid layer as rigid domains. This assumption corresponds to neglecting the terms $\partial v_i/\partial t$ $(i=1,2)$ in equation (3.2.6). At the same time (3.2.12) assumes the form

$$\ddot{w}_0 - pw_0 + p = 0; \quad -\infty < x < -1;$$

$$\ddot{w}_0 = 0; \quad -1 < x < y(\tau); \tag{3.2.22}$$

$$\ddot{w}_2 = 0; \quad y(\tau) < x < 0.$$

Integration and use of the boundary condition gives

$$w_0 = 1 + \frac{\nu_1\nu_2 \exp[(x+1)\sqrt{p}]}{z\sqrt{p} - \nu_1\nu_2};$$

$$w_1 = \frac{[(1-\nu_2)y + \nu_2 x]\sqrt{p}}{z\sqrt{p} - \nu_1\nu_2}; \quad w_2 = \frac{x\sqrt{p}}{z\sqrt{p} - \nu_1\nu_2}, \tag{3.2.23}$$

where

$$\nu_i = \frac{\lambda_i}{\lambda_{i+1}} \quad (i = 1, 2); \quad z = (1-\nu_2)y(\tau) - \nu_2. \tag{3.2.24}$$

Using the known relations [30]

$$\sqrt{p}(\sqrt{p}+\beta)^{-1} \div \exp(\beta^2 t)\operatorname{erfc}(\beta\sqrt{t});$$

$$(\sqrt{p}+\beta)^{-1}\exp(-\alpha\sqrt{p})$$

$$\div \frac{1}{\beta}\operatorname{erfc}\frac{\alpha}{2\sqrt{t}} - \frac{1}{\beta}\exp(\alpha\beta + \beta^2 t)\operatorname{erfc}\left(\frac{\alpha}{2\sqrt{t}} + \beta\sqrt{t}\right) \tag{3.2.25}$$

we find that

$$v_0(x,t;\tau) = 1 + \operatorname{erfc}\frac{|x+1|}{2\sqrt{t}} - \operatorname{erfc}\left(\frac{|x+1|}{2\sqrt{t}} + \frac{\nu_1\nu_2}{|z|}\sqrt{t}\right)$$

$$\cdot \exp\left(\frac{\nu_1\nu_2(x+1)}{z} + \frac{\nu_1^2\nu_2^2}{z^2}t\right); \tag{3.2.26}$$

$$v_1(x,t;\tau) = z^{-1}[(1-\nu_2)y(\tau) + \nu_2 x]\exp\left(\frac{\nu_1^2\nu_2^2}{z^2}t\right)\operatorname{erfc}\left(\frac{\nu_1\nu_2}{|z|}\sqrt{t}\right);$$

$$v_2(x,t;\tau) = z^{-1}x\exp\left(\frac{\nu_1^2\nu_2^2}{z^2}t\right)\operatorname{erfc}\left(\frac{\nu_1\nu_2}{|z|}\sqrt{t}\right).$$

Introducing (3.2.26) into (3.2.9) yields the differential equation

$$\dot{y}(t) = -\gamma\nu_2 z^{-1}(t)\exp\left(\frac{\nu_1^2\nu_2^2}{z^2(t)}t\right)\operatorname{erfc}\left(\frac{\nu_1\nu_2}{|z(t)|}\sqrt{t}\right); \quad y(0) = 1, \tag{3.2.27}$$

which, as in equation (3.2.21), can be solved numerically. We note in conclusion that from the method of constructing the equation (3.2.27) we obtain better approximations for large values of t, and, conversely, worse approximations for small values of t, than in equation (3.2.21).

§3. Inverse problems

We again consider filtration of two slightly compressible immiscible fluids through a deformable porous medium. Suppose that the filtration takes place in an unbounded space under the influence of some initial pressure distribution; we find that in dimensionless coordinates the pressure distribution and boundary of phase change in the simplest double-layer case are determined by the equations

$$\frac{\partial^2 p_1}{\partial x^2} = \frac{\partial p_1}{\partial t}; \quad -\infty < x < y(t); \quad t > 0; \tag{3.3.1}$$

$$a^2\frac{\partial^2 p_2}{\partial x^2} = \frac{\partial p_2}{\partial t}; \quad y(t) < x < \infty; \quad t > 0; \tag{3.3.2}$$

$$p_1 = p_2; \quad \lambda\frac{\partial p_1}{\partial x} = \frac{\partial p_2}{\partial x} \text{ for } x = y(t); \quad t > 0; \tag{3.3.3}$$

$$\dot{y}(t) = -\frac{\partial p_1}{\partial x} \text{ for } x = y(t); \quad t > 0; \quad y(0) = 0;^{1)} \tag{3.3.4}$$

$$p_i|_{t=0} = p_{i,0}(x); \quad (i = 1, 2). \tag{3.3.5}$$

If we assume $p_{i,0}(x)$ as given, then the unknowns will be the pressure $p_i(x,t)$ and the phase boundary $x = y(t)$. It is possible, however, to investigate the inverse problem: given $y(t)$, find the initial pressure distribution. A similar kind of problem appears in a somewhat different form in connection with the problem of regulation of the position of the oil-water contact surface for exploitation of oil fields. Below we consider this question in detail. But here we examine the simplest problem of this kind, assuming as given

$$y(t) \equiv \alpha t; \quad \alpha = \text{const} > 0. \tag{3.3.6}$$

[1)] Equation (3.3.4) is obtained from (3.2.7) if we measure pressure in a scale proportional to γ.

We determine the pressure p_1 by the conditions

$$\frac{\partial^2 p_1}{\partial x^2} = \frac{\partial p_1}{\partial t}; \quad -\infty < x < \alpha t; \quad t > 0;$$

$$p_1|_{t=0} = 0; \quad \frac{\partial p_1}{\partial x}\bigg|_{x=\alpha t} = -\alpha; \quad \lim_{x \to -\infty} p_1 = 0. \tag{3.3.7}$$

In this case the pressure will satisfy the conditions

$$\alpha^2 \frac{\partial^2 p_2}{\partial x^2} = \frac{\partial p_2}{\partial t}; \quad \alpha t < x < \infty; \quad t > 0;$$

$$p_2|_{x=\alpha t} = p_1|_{x=\alpha t}; \quad \frac{\partial p_2}{\partial x}\bigg|_{x=\alpha t} = \lambda \frac{\partial p_1}{\partial x}\bigg|_{x=\alpha t} = -\lambda\alpha. \tag{3.3.8}$$

Condition (3.3.4) will then be satisfied automatically.

Thus we have arrived at an "incorrect" Cauchy problem for the heat conduction equation, with given Cauchy conditions on the time-like line.

Problem (3.3.7), (3.3.8) is easily solved. Actually, we first determine p_1. We find

$$g(x, \xi, t, \tau) = E(x - \xi, t - \tau)\left[1 - \exp\left(-\frac{(x - \alpha t)(\xi - \alpha\tau)}{t - \tau}\right)\right] \tag{3.3.9}$$

and

$$q_1(x, t) = \frac{\partial}{\partial x} p_1(x, t). \tag{3.3.10}$$

$g(x, \xi, t, \tau)$ is the Green's function of the first boundary problem for the equation $L_1(p) = 0$ in the domain $-\infty < x < \alpha t$.

From (3.3.7) and (3.3.10) it follows that

$$\frac{\partial^2 q_1}{\partial x^2} = \frac{\partial q_1}{\partial t}; \quad -\infty < x < \alpha t;$$

$$q_1|_{t=0} = 0; \quad q_1|_{x=\alpha t} = -\alpha. \tag{3.3.11}$$

Consequently

$$q_1(x, t) = \alpha \int_0^t \frac{\partial}{\partial \xi} g(x, \alpha\tau, t, \tau)\, d\tau. \tag{3.3.12}$$

Integrating with respect to x and noting that

$$(3.3.13)\quad \frac{\partial}{\partial \xi} g(x,\xi,t,\tau)|_{\xi=\alpha\tau} = -2\frac{\partial}{\partial x} E(x-\alpha\tau, t-\tau) - \alpha E(x-\alpha\tau, t-\tau),$$

we obtain

$$\begin{aligned} p_1(x,t) &= -\alpha \int_x^{-\infty} d\xi_1 \int_0^t \frac{\partial}{\partial \xi} g(\xi_1, \alpha\tau, t, \tau)\, d\tau \\ &= -2\alpha \int_0^t E(x-\alpha\tau, t-\tau)\, d\tau - \frac{\alpha^2}{2}\int_0^t \operatorname{erfc}\left(\frac{\alpha\tau - x}{2\sqrt{t-\tau}}\right) d\tau. \end{aligned} \tag{3.3.14}$$

Hence, in the limit for $x \to \alpha t - 0$, we find that

$$\begin{aligned} p_1(\alpha t, t) &\equiv f(t) \\ &= -\frac{\alpha}{\sqrt{\pi}} \int_0^t \frac{\exp(-\alpha^2\tau/4)}{\sqrt{\tau}}\, d\tau - \frac{\alpha^2}{2}\int_0^t \left(1 + \operatorname{erf}\frac{\alpha\sqrt{\tau}}{2}\right) d\tau. \end{aligned} \tag{3.3.15}$$

Now we determine p_2. Because of (3.3.8) and (3.3.15) we have

$$(3.3.16)\quad p_2|_{x=\alpha t} = f(t); \quad \frac{\partial}{\partial x} p_2|_{x=\alpha t} = -\lambda\alpha.$$

We will suppose that p_2 is defined for $\alpha t < x < \infty$. We set

$$(3.3.17)\quad p_2|_{t=0} = \phi(x).$$

Then

$$\begin{aligned} p_2(x,t) &= a^2 \int_0^t f(\tau) \frac{\partial}{\partial \xi} g^*(x, \alpha\tau, t, \tau)\, d\tau \\ &\quad + \int_0^\infty \phi(\xi) g^*(x, \xi, t, 0)\, d\xi = p_2' + p_2''. \end{aligned} \tag{3.3.18}$$

Here $g^*(x,\xi,t,\tau)$ is the Green's function of the first boundary problem for the equation $L_2 p = 0$ on the line $x > \alpha t$. It is obtained from $g(x,\xi,t,\tau)$ by interchanging t with τ, $a^2 t$ with $a^2\tau$, and α with α/a^2.

We examine p_2''. We have

$$(3.3.19)\quad p_2'' = \int_0^\infty \phi(\xi) \frac{\exp\left(-\dfrac{(x-\xi)^2}{4a^2 t}\right)}{2a\sqrt{\pi t}} \left[1 - \exp\left(-\frac{\xi(x-\alpha t)}{a^2 t}\right)\right] d\xi.$$

Differentiating with respect to x for $t > 0$ and passing to the limit for $x \to \alpha t + 0$, we find that

$$\frac{\partial p_2''}{\partial x}\bigg|_{x=at} = \int_0^\infty \frac{\xi\phi(\xi)\exp\left(-\dfrac{(x-\xi)^2}{4a^2t}\right)}{2a^3\sqrt{\pi}t^{3/2}}\,d\xi\bigg|_{x=at}$$

$$\equiv -\frac{2\exp\left(-\dfrac{\alpha^2 t}{4a^2}\right)}{\sqrt{\pi}}\int_0^\infty \phi(\xi)\exp\left(\frac{\alpha\xi}{2a^2}\right)\frac{\partial}{\partial\xi}\left(\frac{\exp\left(-\dfrac{\xi^2}{4a^2t}\right)}{2a\sqrt{t}}\right)d\xi. \tag{3.3.20}$$

Assume that for $t = \text{const} > 0$

$$\phi(0) = 0; \quad \lim_{\xi\to\infty}\frac{\phi(\xi)\exp\left(\dfrac{\alpha\xi}{2a^2} - \dfrac{\xi^2}{4a^2t}\right)}{2a\sqrt{t}} = 0. \tag{3.3.21}$$

Integrating (3.3.20) by parts, we obtain

$$\frac{\partial p_2''}{\partial x}\bigg|_{x=at} = \frac{2\exp\left(-\dfrac{\alpha^2 t}{4a^2}\right)}{\sqrt{\pi}}\int_0^\infty \psi(\xi)\exp\left(-\frac{\xi^2}{4a^2t}\right)\frac{d\xi}{2a\sqrt{t}}, \tag{3.3.22}$$

where

$$\psi(x) = \frac{d}{dx}\left(\phi(x)\exp\left(\frac{\alpha x}{2a^2}\right)\right). \tag{3.3.23}$$

We will now examine p_2'. We have

$$p_2' = \int_0^t f(\tau)\frac{x-\alpha t}{2a\sqrt{\pi}(t-\tau)^{3/2}}\cdot\exp\left(-\frac{(x-\alpha\tau)^2}{4a^2(t-\tau)}\right)d\tau,$$

or equivalently

$$p_2' = \int_0^t f(\tau)\left[\frac{\alpha\exp\left(-\dfrac{(x-\alpha\tau)^2}{4a^2(t-\tau)}\right)}{2a\sqrt{\pi}\sqrt{t-\tau}} - \frac{d}{d\tau}\operatorname{erfc}\frac{x-\alpha\tau}{2a\sqrt{t-\tau}}\right]d\tau. \tag{3.3.24}$$

Integrating by parts and taking into account the fact that $f(0) = 0$, we find that

$$p_2' = \int_0^t\left[\dot{f}(\tau)\operatorname{erfc}\left(\frac{x-\alpha\tau}{2a\sqrt{t-\tau}}\right) + \alpha f(\tau)E(x-\alpha\tau, a^2(t-\tau))\right]d\tau. \tag{3.3.25}$$

Differentiating with respect to x, we obtain

$$\frac{\partial p_2'}{\partial x} = \int_0^t \left[\alpha f(\tau) \frac{\partial}{\partial x} E(x - \alpha\tau, a^2(t-\tau)) - 2\dot{f}(\tau) E(x - \alpha\tau, a^2(t-\tau))\right] d\tau. \tag{3.3.26}$$

Now, taking the limit for $x \to \alpha t + 0$ and using the theorem on the discontinuity of the heat potential for a double layer, we find that

$$\frac{\partial p_2'}{\partial x}\bigg|_{x=\alpha t} = -\frac{\alpha f(t)}{2a^2} - \int_0^t \left(f(\tau) + \frac{\alpha^2 f(\tau)}{4a^2}\right) \frac{\exp\left(-\frac{\alpha^2(t-\tau)}{4a^2}\right)}{a\sqrt{\pi(t-\tau)}}\, d\tau. \tag{3.3.27}$$

Combining (3.3.22) and (3.3.27) and using (3.3.18) and (3.3.15), we obtain

$$\int_0^\infty \psi(\xi) \frac{\exp\left(-\frac{\xi^2}{4a^2 t}\right)}{2a\sqrt{t}}\, d\xi = \sum_{i=1}^{3} F_i(t) \equiv F(t), \tag{3.3.28}$$

where

$$F_1(t) = \frac{\alpha\sqrt{\pi}}{2}\left(\frac{f(t)}{2a^2} - \lambda\right) \exp\left(\frac{\alpha^2 t}{4a^2}\right);$$

$$F_2(t) = \int_0^t \dot{f}(\tau) \exp\left(\frac{\alpha^2\tau}{4a^2}\right) \frac{d\tau}{2a\sqrt{t-\tau}}; \tag{3.3.29}$$

$$F_3(t) = \int_0^t \alpha^2 f(\tau) \exp\left(\frac{\alpha^2\tau}{4a^2}\right) \frac{d\tau}{8a^3\sqrt{t-\tau}}.$$

We thus obtain

$$\psi(u) = \sum_{n=0}^{\infty} \alpha_n u^n; \quad F(t) = \sum_{n=0}^{\infty} a_n t^{n/2}. \tag{3.3.30}$$

It is easily seen that if the series (3.3.30) converges uniformly, then

$$\alpha_n = \frac{2a_n}{(2a)^n \Gamma\left(\frac{n+1}{2}\right)} \qquad (n = 0, 1, 2, \cdots). \tag{3.3.31}$$

Comparing (3.3.30) and (3.3.23), we obtain

$$\phi(x) = \exp\left(-\frac{\alpha x}{2a^2}\right) \sum_{n=0}^{\infty} \beta_n x^{n+1}, \tag{3.3.32}$$

where

$$\beta_n = \frac{2a_n}{(2a)^n \Gamma\left(\frac{n+1}{2}\right)(n+1)}. \tag{3.3.33}$$

Thus in order to determine $\phi(x)$ it remains for us to find the coefficients a_n of the expansion $F(t)$. With this aim in mind we set

$$f(t) = \sum_{n=0}^{\infty} \rho_n t^{n/2}. \tag{3.3.34}$$

By virtue of (3.3.15) we have

$$f(t) = -\frac{\alpha^2 t}{2} - \frac{4}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\alpha\sqrt{t}}{2}\right)^{2n+1} \frac{1}{2n+1}$$

$$-\frac{2\alpha^2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\alpha}{2}\right)^{2n+1} \frac{t^{(2n+3)/2}}{(2n+1)(2n+3)}. \tag{3.3.35}$$

Comparing (3.3.35) and (3.3.34), we see that

$$\rho_0 = 0; \quad \rho_1 = -\frac{2\alpha}{\sqrt{\pi}}; \quad \rho_2 = -\frac{\alpha^2}{2};$$

$$\rho_{2n} = 0 \text{ for } n > 2; \tag{3.3.36}$$

$$\rho_{2n+1} = \frac{4}{\sqrt{\pi}} \frac{(-1)^n}{n!} \left(\frac{\alpha}{2}\right)^{2n+1} \frac{1}{4n^2-1} \text{ for } n \geqq 1.$$

We now set

$$\gamma_m = \begin{cases} 0 & \text{for } m = 2k+1, \\ \left(\frac{\alpha}{2a}\right)^{2k} \frac{1}{k!} & \text{for } m = 2k, \end{cases} \tag{3.3.37}$$

so that

$$\exp\frac{\alpha^2 t}{4a^2} = \sum_{m=0}^{\infty} \gamma_m t^{m/2}. \tag{3.3.38}$$

Introducing (3.3.38) and (3.3.34) into (3.3.39), and taking into account the fact that $\rho_0 = 0$, we find that

$$F_1(t) = \frac{1}{2a}\sum_{n=0}^{\infty} t^{n/2}\sum_{k=0}^{n}\gamma_{n-k}c_k;$$

(3.3.39)
$$F_2(t) = \frac{1}{2a}\sum_{n=0}^{\infty} t^{n/2}\sum_{k=0}^{n}\gamma_{n-k}d_k';$$

$$F_3(t) = \frac{1}{2a}\sum_{n=0}^{\infty} t^{n/2}\sum_{k=0}^{n}\gamma_{n-k}d_k'',$$

where

$$c_0 = -\lambda\alpha a\sqrt{\pi}; \quad c_n = \frac{\alpha\rho_n\sqrt{\pi}}{2a}; \quad n > 0;$$

(3.3.39*)
$$d_k' = \frac{1}{2}\,\mathrm{B}\left(\frac{n+1}{2},\frac{1}{2}\right)(k+1)\rho_{k+1}; \quad k \geqq 0;$$

$$d_k'' = \frac{1}{2}\,\mathrm{B}\left(\frac{n+1}{2},\frac{1}{2}\right)\frac{\alpha^2}{2a^2}\rho_{k-1}; \quad k \geqq 0.$$

Therefore, using (3.3.29) and (3.3.30), we obtain

(3.3.40)
$$a_n = \frac{1}{2a}\sum_{k=0}^{n}\gamma_{n-k}\left[c_k + \frac{1}{2}\,\mathrm{B}\left(\frac{n+1}{2},\frac{1}{2}\right)d_k\right],$$

where

(3.3.40*)
$$d_k = \frac{\alpha^2}{2a^2}\rho_{k-1} + (k+1)\rho_{k+1}; \quad (k = 0,1,2,\cdots)$$

and $\mathrm{B}(p,q)$ is the Euler function of first order.

It is easily seen that the sequence $\{a_n\}$ is uniformly bounded; this is because the function $F(t)$ is majorized by the function $\sqrt{t}$. Hence $\phi(x)$ is majorized by a function of the form

(3.3.41)
$$Mx\left(1 - \frac{x}{2a}\right)\exp\left\{-\frac{\alpha x}{2a^2} + \frac{x^2}{4a^2}\right\}; \quad M = \text{const} > 0.$$

But this implies that for sufficiently small $t > 0$ the equation (3.3.1) is automatically satisfied. Thus every operation which is valid for constant $\phi(x)$ remains valid for variable $\phi(x)$; hence the nonrigorous construction of the solution is justified.

It can easily be shown that the solution is unique. In fact, p_1 is the solution of the first boundary problem for the equation of Fourier with an a priori known boundary. Consequently p_1 is uniquely determined. Hence from (3.3.18) and (3.3.20) it follows that the uniqueness of the solution will be proved if the equation

$$\int_0^\infty \frac{\xi\phi(\xi)}{2a^3\sqrt{\pi}t^{3/2}} \exp\left\{-\frac{(\alpha t-\xi)^2}{4a^2t}\right\} d\xi \equiv 0 \tag{3.3.42}$$

implies $\phi(x) \equiv 0$. But this is a well-known fact from the theory of Laplace transforms.

The example investigated is important only as an illustration of the fact that in particular cases the exact solution of an inverse problem of Stefan type can be constructed. Our example has no useful physical application, at least not to the oil production problem here discussed. This is because the determination of the pressure distribution in the halfspace $x > 0$ for $t = 0$ from a given pressure distribution in the halfspace $x < 0$ is physically impossible. Nevertheless, the same class of problems, inverse to the problems of Stefan type, have importance in applications when they are posed as problems of control, which may refer to the problem of filtration, as well as to the thermal problem.

As an example of a simple problem of control we examine the following one-dimensional problem.

Consider a one-dimensional process of displacement of oil by water from a homogeneous stratum in an elastic regime. We assume that the sampling is taken from a shaft and that a given pressure gradient is maintained at the point of sampling. At the initial moment the pressure in the entire stratum is constant. (It will be assumed that the initial pressure is zero, which merely corresponds to a shift in the zero-point of reference.) It is required to find the position and cost of a delivery shaft, such that the water-oil contact, given by $x = y(t)$, progresses according to a preassigned law.

We will assume that the permeability and piezoconductivity in the intermediate zone and in the water zone are the same. We regard the delivery shaft as a source and make use of dimensionless variables analogous to those introduced in §2; we then find that the problem reduces to one of determining a function $q(t)$ and some $L < 0$, satisfying the conditions

(3.3.43) $$p_1(x,t) = p_0(x,t) + P_0(x,t);$$

(3.3.44) $$\frac{\partial^2 p_0}{\partial x} = \frac{\partial p_0}{\partial t}; \quad -\infty < x < y(t); \quad t > 0;$$

(3.3.45) $$a^2 \frac{\partial^2 p_2}{\partial x^2} = \frac{\partial p_2}{\partial t}; \quad y(t) < x < 0; \quad t > 0;$$

(3.3.46) $$p_0(x,0) = p_2(x,0) = 0; \quad p_2(0,t) = P(t);$$

(3.3.47) $$p_1 = p_2; \quad \lambda \frac{\partial p_1}{\partial x} = \frac{\partial p_2}{\partial x} \text{ for } x = y(t); \quad t > 0;$$

(3.3.48) $$\dot{y}(t) = -\frac{\partial p_2}{\partial x}\bigg|_{x=y(t)}; \quad t > 0; \quad y(0) = -l < 0.$$

Here

(3.3.49) $$P_0(x,t) = \int_0^t q(\tau) E(x-l, t-\tau)\, d\tau$$

is an influence function acting continuously at the source with intensity $q(t)$, and $y(t)$ is a known smooth monotonically increasing function.

It is evident that a solution to the inverse problem can be found only for $L = y(0) = -l$. In fact, in the opposite case the equation

(3.3.50) $$\dot{y}(0) = -\lambda \frac{\partial}{\partial x} p_0(x,0)\,|_{x=y(0)} = 0,$$

which is implied by the conditions of the problem, is impossible for an arbitrarily given $y(t)$. Thus we will naturally require that $L = -l$.

For a given $y(t)$ the pressure p_1 can be assumed as known. For its determination it is sufficient to construct a Green's function for the first boundary problem of the equation $L_a p = 0$ in the domain $y(t) < x < 0$. Also, we may assume that the values of p_1 and $\partial p_1/\partial x$ are known on the contour $x = y(t)$, as well as the Green's function of the first boundary problem for the equation $L_1 p = 0$ in the domain $-\infty < x < y(t)$. Then the problem (3.3.43)—(3.3.48) can be given a second formulation. Namely, suppose we are given the functions $f(t)$, $\phi(t)$ and $y(t)$ $(y(0) = 0)$. Let $g(x,\xi,t,\tau)$ be the Green's function for the first boundary problem of the equation $L_1(p) = 0$ in the domain $-\infty < x < y(t)$. We are required to find $q(t)$ such that if

$$p_1(x,t) = \int_0^t q(\tau) g(x,0,t,\tau)\, d\tau - \int_0^t f(\tau) \frac{\partial}{\partial \xi} g(x, y(\tau), t, \tau)\, d\tau, \tag{3.3.51}$$

then

$$\lim_{x \to y(t)-0} \frac{\partial}{\partial x} p_1(x,t) = \phi(t). \tag{3.3.52}$$

Thus the investigation of the control problem, and together with it the entire class of similar control problems for filtration or heat conduction, is reduced to the problem of solving an integral equation of Volterra type of the first kind

$$\int_0^t q(\tau) \frac{\partial}{\partial x} g(y(t), 0, t, \tau)\, d\tau = F(t). \tag{3.3.53}$$

From the properties of the Green's function it follows that

$$\frac{\partial}{\partial x} g(y(t), 0, t, t) = 0; \quad t > 0. \tag{3.3.54}$$

This shows that the problem is related to the class of problems which are not correct, in the terminology of A. N. Tihonov [175]. A general theory of the solution of such problems has been developed by Tihonov and his students. However, in a concrete case, as in the theory of the problem inverse to the Stefan problem, it has not been investigated. Aside from the above discussion, taken from [139], the inverse problem of Stefan type has been examined only in the papers [97], [98] of G. A. Martynov, mentioned above, and by approximation methods. Thus the inverse Stefan problem, when considered as a problem of control, remains without solution.[2)]

[2)] An examination of a general formulation of the inverse problem lies outside the framework of the problems analyzed in the second part of this book. For this reason we have limited our remarks at this point to those above.

Chapter IV

Some Problems of Mechanics of Continuous Media that Reduce to Problems of Stefan Type

§1. A problem of convection arising from crystallization of a supercooled melt [18]

In Chapters I and II, when investigating heat and thermal diffusion processes, we always disregarded jumps of density induced by phase changes. However, such jumps are unavoidable, and cause the appearance of convection and flow in the liquid phase even when there is no applied force present.

We will now investigate the crystallization of a supercooled melt situated in contact with its solid phase, following the work of P. L. Chambré [18]; in addition, we disregard the pressure gradient and viscous dissipation of energy, but take into account the density jump under the change of phase.

Assume that at the time $t > 0$ the solid phase occupies the domain $-\infty < x < y(t)$, and the liquid phase $y(t) < x < \infty$. Let the index $i = 1$ refer to the liquid phase, and $i = 2$ to the solid phase. We consider the process as one-dimensional; and, using the usual notation, we obtain the equations of motion and energy in the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}; \quad y(t) < x < \infty; \quad t > 0;$$

$$\frac{\partial T_2}{\partial t} = a_2^2\frac{\partial^2 T_2}{\partial x^2}; \quad -\infty < x < y(t); \quad t > 0; \tag{4.1.1}$$

$$\frac{\partial T_1}{\partial t} + u\frac{\partial T_1}{\partial x} = a_1^2\frac{\partial^2 T_1}{\partial x^2}; \quad y(t) < x < \infty; \quad t > 0.$$

The initial conditions and the conditions at infinity are

$$u|_{t=0} = 0; \quad T_i|_{t=0} = T_{i,0}, \quad i = 1, 2;$$

$$\lim_{x\to\infty} u = 0; \quad \lim_{|x|\to\infty} T_i = T_{i,0}, \quad i = 1, 2. \tag{4.1.2}$$

At the crystallization front the usual conditions of continuity of tempera-

ture and heat balance take the form

$$T_1 = T_2 = T_s \text{ for } x = y(t); \quad t > 0; \tag{4.1.3}$$

$$\lambda\rho_2\dot{y}(t) = \left(k_2\frac{\partial T_2}{\partial x} - k_1\frac{\partial T_1}{\partial x}\right) \text{ for } x = y(t); \quad t > 0; \quad y(0) = 0.^{1)} \tag{4.1.4}$$

We turn to the condition of material balance at the phase boundary. Suppose that in time dt the phase boundary moves a distance dy into the liquid domain. Then in a unit cross-section the mass of the solid material formed is equal to

$$dM = \rho_2 dy. \tag{4.1.6$_1$}$$

In the volume dy undergoing a phase transition the mass of the melt is

$$dM_1 = \rho_1 dy. \tag{4.1.6$_2$}$$

Finally, from the volume dy within the liquid phase a mass is generated which is equal to

$$dM_2 = \rho_1 u(y(t), t)\, dt. \tag{4.1.6$_3$}$$

Thus $dM_1 - dM = dM_2$, which shows that

$$u(y(t), t) = -\epsilon\dot{y}(t), \tag{4.1.6}$$

where

$$\epsilon = \frac{\rho_2 - \rho_1}{\rho_1}. \tag{4.1.7}$$

Problem (4.1.1)—(4.1.6) admits a similarity solution which we seek in the form

$$\begin{gathered} y = 2\beta\sqrt{\nu t}; \quad u = 2\sqrt{\frac{\nu}{\pi t}}f(\eta); \quad T_i = T_i(\eta); \\ \eta = \frac{x}{2\sqrt{\nu t}} \qquad (i = 1, 2). \end{gathered} \tag{4.1.8}$$

Substituting (4.1.8) into (4.1.1)—(4.1.6), we find that β, $f(\eta)$ and $T_i(\eta)$

[1)] We include in the factor λ the energy of dissipation by the isothermal work of expansion under the phase change. If L denotes the latent heat of the phase transition under constant volume, and c_1 and c_2 denote the specific heats of the liquid and solid materials respectively for the crystallization temperature T_s, then obviously

$$\lambda\rho_2\dot{y}(t) = \rho_2(L + c_2T_s)\dot{y}(t) - \rho_1c_1T_s[\dot{y}(t) - u(y(t), t)]. \tag{4.1.5}$$

are to be determined from the conditions

$$\ddot{f}(\eta) - \frac{4}{\sqrt{\pi}} \dot{f}(\eta) f(\eta) + 2(f(\eta) + \eta \dot{f}(\eta)) = 0; \tag{4.1.9}$$

$$\ddot{T}_1(\eta) + \left[2\sigma_1 \eta - \frac{4}{\sqrt{\pi}} \sigma_1 f(\eta) \right] \dot{T}_1(\eta) = 0; \tag{4.1.10}$$

$$\ddot{T}_2(\eta) + 2\sigma_2 \eta \dot{T}_2(\eta) = 0; \tag{4.1.11}$$

$$T_i(\beta) = T_s \quad (i = 1,2); \quad \lim_{|\eta| \to \infty} T_i(\eta) = T_{i,0} \quad (i = 1,2);$$

$$f(\beta) = -\frac{\sqrt{\pi}}{2} \epsilon\beta; \quad \lim_{\eta \to \infty} \dot{f}(\eta) = 0; \quad \lim_{\eta \to \infty} f(\eta) = 0; \tag{4.1.12}$$

$$-\mu\beta = [\dot{T}_1(\eta) - \gamma \dot{T}_2(\eta)]_{\eta=\beta}. \tag{4.1.13}$$

Here

$$\sigma_1 = \frac{\nu}{a_1^2}; \quad \sigma_2 = \frac{\nu}{a_2^2}; \quad \gamma = \frac{k_2}{k_1}; \quad \mu = \frac{2\nu\lambda\rho_2}{k_1}. \tag{4.1.14}$$

Integrating (4.1.9) and using (4.1.12), we easily find that

$$f(\eta) = -\frac{\epsilon\beta \exp[-(\eta^2 - \beta^2)]}{2/\sqrt{\pi} + \epsilon\beta(\operatorname{erf}\eta - \operatorname{erf}\beta)\exp\beta^2}. \tag{4.1.15}$$

Substituting this into (4.1.10), integrating and using the condition (4.1.12), we find that

$$T_1(\eta) = T_{1,0} + (T_s - T_{1,0}) R(\eta, \beta), \tag{4.1.16}$$

where

$$R(\eta,\beta) = \frac{\displaystyle\int_\eta^\infty \exp(-\sigma_1 t^2) \left(\frac{2}{\sqrt{\pi}} + \epsilon\beta \exp\beta^2 [\operatorname{erf} t - \operatorname{erf}\beta] \right)^{-2\sigma_1} dt}{\displaystyle\int_\beta^\infty \exp(-\sigma_1 t^2) \left(\frac{2}{\sqrt{\pi}} + \epsilon\beta \exp\beta^2 [\operatorname{erf} t - \operatorname{erf}\beta] \right)^{-2\sigma_1} dt}. \tag{4.1.16*}$$

Finally, integrating (4.1.11) and using (4.1.12) gives

$$T_2(\eta) = T_{2,0} + (T_s - T_{2,0}) \frac{1 + \operatorname{erf}\eta\sqrt{\sigma_2}}{1 + \operatorname{erf}\beta\sqrt{\sigma_2}}. \tag{4.1.17}$$

Substituting T_1 and T_2 from (4.1.16) and (4.1.17) into (4.1.13) yields a condition by which we can determine β. We confine our investigation to the simplest case

(4.1.18) $$\sigma_1 = 1,$$

which is satisfied for a gas. In this case, after integration the equation (4.1.16) assumes the form

(4.1.19) $$T_1 = T_{1,0} + (T_s - T_{1,0}) \frac{(2/\sqrt{\pi}) \operatorname{erfc} \eta}{\operatorname{erfc} \beta [2/\sqrt{\pi} + \epsilon \beta \exp \beta^2 (\operatorname{erf} \eta - \operatorname{erf} \beta)]}.$$

Hence from (4.1.17) and (4.1.13) we obtain

(4.1.20) $$[\mu - \epsilon(T_s - T_{1,0})]\beta = \frac{2}{\sqrt{\pi}} \left[(T_s - T_{1,0}) \frac{\exp(-\beta^2)}{\operatorname{erfc} \beta} + \gamma \sqrt{\sigma_2} \frac{(T_s - T_{2,0}) \exp(-\sigma_2 \beta^2)}{1 + \operatorname{erf} \beta \sqrt{\sigma_2}} \right] \equiv f(\beta).$$

We note that

(4.1.21) $$\mu > 0; \quad -1 < \epsilon < \infty; \quad \gamma \sqrt{\sigma_2} > 0; \quad \sigma_2 > 0.$$

A thorough investigation of equation (4.1.20) for every possible choice of the known parameters of the process is difficult. We note merely that for $\epsilon \geqq 0$ (i.e. when crystallization is accompanied by an increase in the density of the material, which occurs in the majority of physically realizable cases) the existence of a unique root of equation (4.1.20) is obvious if at the initial moment the solid phase is supercooled and the liquid phase superheated, i.e. if

(4.1.22) $$T_{2,0} < T_s < T_{1,0}.$$

In fact, in this case $f(\beta)$ is monotonically decreasing, with

$$\lim_{\beta \to -\infty} f(\beta) = -\infty; \quad \lim_{\beta \to +\infty} f(\beta) = -\infty,$$

while at the same time the left side of (4.1.20) is monotonically increasing.

We will investigate the work of P. L. Chambré in detail for the case[2)]

(4.1.23) $$T_{2,0} = T_s.$$

In this case (4.1.20) can be written in the form

Substituting (4.1.8) into (4.1.1)—(4.1.6), we find that β, $f(\eta)$ and $T_i(\eta)$

[2)] Chambré assumed that the conductivity of the solid phase was infinitely large. In this case $T_2 = T_s$. It is obvious that such an assumption is not necessary if we assume that at the initial moment the solid phase is found at the temperature of crystallization; in this case the problem reduces to one of supercooling of the melt.

$$\frac{T_s - T_{1,0}}{\Theta} = \frac{2(1+\epsilon)\beta \exp\beta^2 \operatorname{erfc}\beta}{2/\sqrt{\pi} + \epsilon\beta \exp\beta^2 \operatorname{erfc}\beta} \equiv F(\epsilon, \beta), \tag{4.1.24}$$

where

$$\Theta = \frac{L}{c_1} + T_s\left(\frac{c_2}{c_1} - 1\right).^{3)} \tag{4.1.25}$$

For arbitrary fixed $\beta \geqq 0$ F is an increasing function of ϵ, varying from zero to two. This shows that if $(T_s - T_{1,0})/\Theta > 2$, then a solution to equation (4.1.24), and hence a solution to our problem, cannot exist. By the same token a limit on the possible depth of the supercooled melt is established. Conversely, if we are given the speed β of crystallization, it is possible to determine the corresponding depth of the supercooled melt.

In particular, we note that for $\epsilon = 0$, i.e. for equal densities of solid and liquid phases and hence in the absence of convection, we have

$$0 < \frac{T_s - T_{1,0}}{\Theta} = \sqrt{\pi}\beta \exp\beta^2 \operatorname{erfc}\beta < 1, \tag{4.1.26}$$

which shows that in this case the maximum possible depth of supercooling of the melt is to a point which is defined by the adiabatic cooling of the melt.

§2. High speed flow of a solid body in a viscous incompressible fluid [56], [137].

Let there be given a solid body occupying the halfspace $-\infty < x_1 < l_1$ bounded by the surface $x_1 = l_1$, and a viscous incompressible fluid occupying the halfspace $x_1 > l_1$. Assume that this body is moving with velocity $v_i = \text{const.}$ parallel to the plane $x_1 = l_1$; consequently, due to the viscous dissipation of energy, the temperature at the boundary $x_1 = l_1$ at time $t_1 = 0$ attains the melting temperature of the flowing body. Subsequently the process involves three phases: solid, occupying the domain $-\infty < x_1 < y_1(t_1)$; melt, in the domain $y_1(t_1) < x_1 < l_1$; and the flowing liquid, in the domain $l_1 < x_1 < \infty$.[4)] We assume that the

[3)] Equation (4.1.24) follows at once from (4.1.20) if we note that by virtue of (4.1.5) and (4.1.14) $\mu = 2\delta_1(1+\epsilon)(Lc_1^{-1} + (c_2 - c_1)C_1^{-1}T_s)$ and that by assumption $\sigma_1 = 1$.

[4)] If the flowing liquid is a gas, then the melt will vaporize and thus the boundary of the melt and the flowing liquid will vary with time and be subject to determination. For the sake of simplicity we neglect this case in our discussion.

state of the system for $t_1 < 0$ is known, as are the temperature distribution and velocity for $t_1 = 0$.

We will assign the index $i = 1$ to the solid phase, $i = 2$ to the melt, and $i = 3$ to the flowing liquid. Let ϑ_i, w_i, k_i^*, $c_{p,i}$, ρ_i, a_i^{2*} and ν_i denote the corresponding temperature, velocity, coefficient of heat conduction, specific heat for constant pressure, density, coefficient of conductivity and kinematic viscosity of the ith phase. Moreover, let λ^* denote the latent heat of the phase change per unit mass. Let k_i^*, $c_{p,i}$, ρ_i, a_i^{2*}, ν_i and λ^* be constants, $\rho_1 = \rho_2 = \rho$. Let $\psi_{i,1}(x_1)$ be the initial temperature of the ith phase and $\phi_{i,1}(x_1)$ the initial velocity of this phase. Assume that

$$T = \max\{|\psi_{i,1}(x_1)|, \quad i = 1,2,3, \quad -\infty < x < \infty\} \tag{4.2.1}$$

and let us introduce the dimensionless variables

$$v_i = \frac{w_i}{v_0}; \quad u_i = \frac{\vartheta_i}{T}; \quad \psi_i = \frac{\psi_{i,1}}{T}; \quad \phi_i = \frac{\phi_{i,1}}{v_0};$$

$$x = \frac{x_1}{l_1}; \quad t = \frac{a_1^{2*} t_1}{l_1^2}; \quad y(t) = \frac{y_1(t_1)}{l_1}; \quad a_i^2 = \frac{a_i^{2*}}{a_1^{2*}}; \quad b_i^2 = \frac{\nu_i}{a_1^{2*}}; \tag{4.2.2}$$

$$\gamma_i^2 = \frac{\nu_i v_0^2}{c_{p,i} a_1^{2*} T}; \quad \lambda_i = \rho_i \nu_i; \quad k_i = \frac{k_i^* T}{\lambda^* \rho a_1^{2*}} \quad (i = 1,2,3).$$

The indices referring to the liquid phase are omitted.

We are required to determine u_1, u, v and y from the conditions

$$\frac{\partial^2 u_1}{\partial x^2} = \frac{\partial u_1}{\partial t} \text{ for } -\infty < x < y(t); \quad u_1|_{t=0} = \psi_1(x); \quad u_1|_{x=y(t)} = 0; \tag{4.2.3$_1$}$$

$$b^2 \frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t} \text{ for } y(t) < x < \infty; \quad x \neq 1; \quad v|_{t=0} = \phi(x); \quad v|_{x=y(t)} = 0;$$
$$v \text{ and } \lambda \frac{\partial v}{\partial x} \text{ are continuous for } x = 1; \quad t > 0; \tag{4.2.3$_2$}$$

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} - \gamma^2 \left(\frac{\partial v}{\partial x}\right)^2 \text{ for } y(t) < x < \infty; \quad x \neq 1; \quad t > 0;$$
$$u|_{x=y(t)} = 0; \quad u|_{t=0} = \psi(x); \tag{4.2.3$_3$}$$
$$u \text{ and } k \frac{\partial u}{\partial x} \text{ are continuous for } x = 1; \quad t > 0;$$

$$\dot{y}(t) = \left(k_1 \frac{\partial u_1}{\partial x} - k_2 \frac{\partial u}{\partial x}\right)\Bigg|_{x=y(t)}; \quad y(0) = 1. \tag{4.2.3$_4$}$$

We assume that

$$\psi_1(x) \leqq 0; \quad \psi(x) \geqq 0. \tag{4.2.3$_5$}$$

It is possible to base a numerical solution for this general formulation of the problem on the method of construction of a solution to the Stefan problem which will be examined in Part 2, or on approximation, for example, by means of the quasi-stationary approximations of L. S. Leĭbenzon. One frequently encounters the case

$$\phi = \text{const}; \quad \psi_1 = \text{const} > 0; \quad \psi = \text{const} > 0; \quad y(0) = 1, \tag{4.2.4}$$

which was investigated by S. S. Grigorjan [56], and for which the problem admits a similarity solution of the form

$$u_1 = u_1(\eta); \quad u = u(\eta); \quad v = v(\eta); \quad y = 2\beta\sqrt{t} + 1; \quad \eta = \frac{x-1}{2\sqrt{t}}. \tag{4.2.5}$$

In fact, substituting this into (4.2.3$_i$) ($i = 1, \cdots, 4$) gives us

$$\begin{gathered} \ddot{u}_1 + 2\eta\dot{u}_1 = 0; \quad -\infty < \eta < \beta < 0; \\ a^2\ddot{u} + 2\eta\dot{u} + \gamma^2\dot{v}^2 = 0; \quad \beta < \eta < \infty; \quad \eta \neq 0; \\ b^2\ddot{v} + 2\eta\dot{v} = 0; \quad \beta < \eta < \infty; \quad \eta \neq 0. \end{gathered} \tag{4.2.6}$$

$$\begin{gathered} u_1 = u = v = 0 \text{ for } \eta = \beta; \\ \lim_{\eta \to -\infty} u_1 = \psi_1; \quad \lim_{\eta \to +\infty} u = \psi; \quad \lim_{\eta \to +\infty} v = \phi; \\ u, v, k\frac{du}{d\eta}, \lambda\frac{dv}{d\eta} \text{ are continuous for } \eta = 0, \end{gathered} \tag{4.2.7}$$

$$\beta = \tfrac{1}{2}(k_1\dot{u}_1 - k_2\dot{u})_{\eta-\beta}. \tag{4.2.8}$$

Integrating (4.2.6) and using the conditions (4.2.7), we find that

$$u_1 = \psi_1\left(1 - \frac{1 + \operatorname{erf}\eta}{1 + \operatorname{erf}\beta}\right); \quad -\infty < \eta < \beta; \tag{4.2.9}$$

$$v = \begin{cases} \phi\left(\operatorname{erf}\dfrac{\eta}{b_2} - \operatorname{erf}\dfrac{\beta}{b_2}\right)\left(\delta - \operatorname{erf}\dfrac{\beta}{b_2}\right)^{-1}; & \beta < \eta < 0; \\ \phi\left[1 - \delta \operatorname{erfc}\dfrac{\eta}{b_3}\left(\delta - \operatorname{erf}\dfrac{\beta}{b_2}\right)^{-1}\right]; & 0 < \eta < \infty; \end{cases} \tag{4.2.10}$$

$$u = \begin{cases} A\left(\operatorname{erf}\frac{\eta}{a_2} - \operatorname{erf}\frac{\beta}{a_2}\right) + \frac{\sqrt{\pi}}{2}\cdot\frac{\gamma_2^2}{a_2}\int_\beta^\eta \exp\left(\frac{t^2}{a_2^2}\right)\dot{v}^2(t)\left(\operatorname{erf}\frac{t}{a_2} - \operatorname{erf}\frac{\eta}{a_2}\right)dt \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \beta < \eta < 0; \\ \psi - B\operatorname{erfc}\frac{\eta}{a_3} + \frac{\sqrt{\pi}}{2}\cdot\frac{\gamma_3^2}{a_3}\int_\eta^\infty \exp\left(\frac{t^2}{a_3^2}\right)\dot{v}^2(t)\left(\operatorname{erfc}\frac{\eta}{a_3} - \operatorname{erfc}\frac{t}{a_3}\right)dt \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } 0 < \eta < \infty. \end{cases}$$

(4.2.11)

Here

$$A = \left(\mu - \operatorname{erf}\frac{\beta}{a_2}\right)^{-1}\left(\frac{\sqrt{\pi}}{2}\int_\beta^\infty \exp\left(\frac{t^2}{a^2}\right)\dot{v}^2(t)\left(g - \frac{\gamma^2}{a}\operatorname{erf}\frac{t}{a}\right)dt + \psi\right);$$

(4.2.12)

$$B = \mu A - \frac{\sqrt{\pi}}{2}\int_\beta^\infty \exp\left(\frac{t^2}{a^2}\right)\dot{v}^2(t)g\,dt;$$

$$\delta = \frac{\lambda_2 b_3}{\lambda_3 b_2};\quad \mu = \frac{k_2 a_3}{k_3 a_2};\quad g(t) = \begin{cases} \mu\frac{\gamma_2^2}{a_2} \text{ for } \beta < t < 0; \\ \frac{\gamma_3^2}{a_3} \text{ for } 0 < t < \infty; \end{cases}$$

(4.2.12*)

$$\gamma(t) = \begin{cases} \gamma_2 \text{ for } \beta < t < 0; \\ \gamma_3 \text{ for } 0 < t < \infty; \end{cases} \qquad a(t) = \begin{cases} a_2 \text{ for } \beta < t < 0; \\ a_3 \text{ for } 0 < t < \infty. \end{cases}$$

Substituting (4.2.11) and (4.2.9) into (4.2.8), we obtain

$$\beta = -\left[\frac{\psi_1 k_1 \exp(-\beta^2)}{\sqrt{\pi}(1 + \operatorname{erf}\beta)} + \frac{A k_2 \exp(-\beta^2/a_2^2)}{a_2\sqrt{\pi}}\right] \equiv F(\beta). \tag{4.2.13}$$

Since $\psi_1 < 0$, we find by using (4.2.12) that

$$\lim_{\beta \to -\infty} F(\beta) = +\infty. \tag{4.2.14}$$

Furthermore,

$$F(0) = -\frac{\psi_1 k_1}{\sqrt{\pi}} - \frac{k_2}{a_2}\cdot\frac{\displaystyle\int_0^\infty \exp\left(\frac{t^2}{a^2}\right)\dot{v}^2(t)\frac{\gamma^2}{a}\operatorname{erfc}\frac{t}{a}\,dt}{2\mu}. \tag{4.2.15}$$

Since $\dot{v}^2(t)$ is proportional to ϕ^2, this implies that $F(0) < 0$ for ϕ^2 sufficiently large. Hence from (4.2.14) it follows that there exists a negative

root of equation (4.2.13) and therefore a solution of the problem under investigation.

The example of Grigorjan is typical of a wide class of problems of ablation of a solid body moving at high velocity in a gas. This problem is particularly important in connection with the development of rocket technology, and is the subject of many discussions in the literature. In a great many cases problems of this type are investigated either in similarity form or by approximation methods. In [137] the problem $(4.2.3_i)$ was solved in a similarity form by reduction to an integral equation, using the method to be described in Part 2 below. Some references are indicated in the Introduction.

§3. Nonstationary flows of a viscous plastic medium

We now pose two Stefan type problems from the theory of nonstationary flows of a viscous plastic medium, which were investigated by A. I. Safrončik [147], [148].

a) Rotation of a rigid cylinder in a viscous plastic medium. We investigate an unbounded Bingham body in which a rigid cylinder is rotating with known angular velocity $\omega(t)$. Let v be the tangential component of the velocity, ρ the density, μ the plasticity viscosity, τ_0 the limiting stress shear, R_0 the radius of the cylinder, r the radial distance, and t the time.

The space $r > R_0$ is divided into two regions: the region $R_0 < r < R(t_1)$ in which the tangential stress exceeds the limiting stress shear and in which therefore the material is itself acting as a viscous fluid, and the domain $R(t_1) < r < \infty$ in which the tangential stress is less than the limiting shear stress and consequently the material is acting like a solid.

The tangential component v of velocity is its only component. In the domain of viscous flow we have the equation

$$\rho \frac{\partial v}{\partial t} = \mu \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) v - \frac{2\tau_0}{r}; \quad R_0 < r < R(t); \quad t > 0. \tag{4.3.1}$$

At the surface of the cylinder we have

$$v|_{r=R_0} = R_0 \omega(t). \tag{4.3.2}$$

At the boundary $r = R(t)$ of the viscous domain the velocity must be zero, and the tangential stress τ must be equal to the yield stress. Since the tangential stress τ for a Bingham body is

(4.3.3) $$\tau - \tau_0 = \pm \mu \frac{\partial v}{\partial n},$$

we find that

(4.3.4) $$v = 0; \quad \frac{\partial v}{\partial r} = 0 \text{ for } r = R(t).$$

We add here the initial conditions

(4.3.5) $$v|_{t=0} = v_0(r); \quad R|_{t=0} = R_1 \geqq R_0,$$

which complete the system of equations of the problem.

We introduce the dimensionless variables

$$x = \frac{r}{R_0}; \quad t_1 = \frac{\mu t}{\rho R_0^2}; \quad y(t_1) = \frac{R(t)}{R_0}; \quad \alpha = \frac{2\tau_0}{\mu\omega_0};$$

(4.3.6) $$u(x, t_1) = \frac{v(r,t)}{R_0\omega_0} - \frac{\alpha}{2} x \ln x;$$

$$u_0(x) = u|_{t_1=0}; \quad \omega_1(t_1) = \frac{\omega(t)}{\omega_0},$$

where ω_0 is a constant with dimensions of angular velocity. Omitting the indices in t_1 and ω_1, we find that u and y obey the system of equations

(4.3.7$_1$) $$\frac{\partial u}{\partial t} = \left(\frac{\partial^2}{\partial x^2} + \frac{1}{x}\frac{\partial}{\partial x} - \frac{1}{x^2}\right) u; \quad 1 < x < y(t); \quad t > 0;$$

(4.3.7$_2$) $$u|_{t=0} = u_0(x); \quad u|_{x=1} = \omega(t) \quad y(0) = 1;$$

(4.3.7$_3$) $$u|_{x=y(t)} = -\frac{\alpha}{2} y(t) \ln y(t); \quad t > 0;$$

(4.3.7$_4$) $$y(t) = \exp\left[-\left(1 + \frac{2}{\alpha}\frac{\partial u}{\partial x}\right)\right]\Bigg|_{x=y(t)}.$$

b) Unsteady viscous plastic flow between plane parallel walls. We now investigate the flow of an incompressible viscous plastic Bingham body between two infinite parallel plane walls under the action of a given pressure gradient. Let (x, y, z) be the rectangular system of coordinates, $\vec{i}, \vec{j}, \vec{k}$ a vector basis, t the time, p the pressure, $\rho = \text{const}$ the density, $v = u\vec{i} + v\vec{j} + w\vec{k}$ the velocity, ν the kinematic viscosity, and τ_0 the limit stress. Now let $y = \pm H$ be the equation of the boundary of the body. Moreover, assume that

(4.3.8) $$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0; \quad v = w = 0; \quad \frac{\partial u}{\partial z} = \frac{\partial u}{\partial x} = 0.$$

The domain $-H < y < H$ is divided into two parts, D_1 and D_2,

$$D_1 = \{-H < y < -y_0(t);\quad y_0(t) < y < H\};$$
$$D_2 = \{-y_0(t) < y < y_0(t)\}, \tag{4.3.9}$$

in which the tangential stress is larger (respectively, smaller) than the limit stress. We refer to the first domain as the domain of viscous flow, and to the second as the core. The core moves as a rigid elastic body, parallel to the planes $y = \pm H$.

In the domain of viscous flow we have the equation[5)]

$$\frac{\partial u}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \nu \frac{\partial^2 u}{\partial y^2};\quad y_0(t) < y < H. \tag{4.3.10}$$

In the core

$$u = u_0(t) \equiv u(y_0(t), t). \tag{4.3.11}$$

From (4.3.10) and (4.3.8) it obviously follows that

$$\left.\frac{1}{\rho}\frac{\partial p}{\partial x}\right| = f(t). \tag{4.3.12}$$

At the boundary $y = H$ we have the usual condition of adhesion, expressed as

$$u|_{y=H} = 0. \tag{4.3.13}$$

At the core boundary the tangential stress is equal to the yield stress, whence from (4.3.3) we conclude that

$$\left.\frac{\partial u}{\partial y}\right|_{y=y_0(t)} = 0. \tag{4.3.14}$$

To obtain the second boundary condition for $y = y_0(t)$ we select a volume

$$\Omega = \{z_0 < z < z_0 + 1;\quad x_0 < x < x_0 + 1;\quad -l < y < l\}, \tag{4.3.15}$$

where $l > y_0(t_0)$ and $t_0 > 0$ is a fixed moment in time. Let t be close to t_0. For the volume Ω we write the principle of conservation of momentum at the time t in the differential form

$$\frac{\partial}{\partial t}\rho \int_{-t}^{t} u(y,t)\,dy = F. \tag{4.3.16}$$

[5)] The flow is symmetric with respect to the plane $y = 0$.

Here F is the resultant of all the volume and surface forces acting on Ω.

Using (4.3.11), (4.3.10) and (4.3.14), and taking into account the symmetry of the flow with respect to the plane $y = 0$, we obtain

$$\frac{\partial}{\partial t}\left(\rho y_0(t)\, u_0(t) + \rho \int_{y_0(t)}^{l} u(y,t)\, dy\right)$$

$$\equiv \rho y_0(t) \frac{du_0}{dt} + \rho\nu \frac{\partial u}{\partial y}\bigg|_{y=l} - \int_{y_0(t)}^{l} \frac{\partial p}{\partial x}\, dy = \frac{1}{2} F. \tag{4.3.17}$$

We now pass to the limit for $t \to t_0$ and $l \to y(t) + 0$. Omitting the index 0 of the time and again using (4.3.14), we obtain

$$\rho y_0(t) \frac{du_0}{dt} = \frac{1}{2} F. \tag{4.3.18}$$

We now calculate the expression F. By the assumptions of the problem we can assume that there are no volume forces. At the faces $z = z_0$ and $z = z_0 + 1$ the components of stress in the direction of the x-axis are equal to zero; on the faces $x = x_0$ and $x = x_0 + 1$ they are equal, respectively, to $p(x_0, t)$ and $p(x_0 + 1, t)$, so that the resultant force operating on these faces is equal to

$$2y_0(t)\,[p(x_0,t) - p(x_0+1,t)] = -2y_0(t)\frac{\partial p}{\partial x} \equiv -2\rho y_0(t) f(t). \tag{4.3.19}$$

Finally, on the face $y = \pm y_0(t)$ the effective stress $-\tau_0$ is such that the resulting force is equal to

$$F = -2\rho y_0(t) f(t) - 2\tau_0. \tag{4.3.20}$$

Substituting this into (4.3.18), we obtain the desired condition

$$y_0(t) \frac{du_0}{dt} = -y_0(t) f(t) - \frac{\tau_0}{\rho}. \tag{4.3.21}$$

If we now adjoin the initial conditions

$$u(y,0) = u_1(y); \quad y_0(0) = y_1, \tag{4.3.22}$$

we complete the formulation of the problem.

In those cases of the problem of unsteady viscous plastic flow which we have investigated, the conditions at the unknown boundary are in principle unlike any of the interface conditions for problems of Stefan type. In fact, in these cases the conditions at the unknown boundary have the form

$$
\begin{aligned}
&y|_{x=y(t)} = \Psi(y(t), t);\\
&\dot{y}(t) = Z\left(u, \frac{\partial u}{\partial x}, x\right)\Big|_{x=y(t)},
\end{aligned}
\tag{4.3.23}
$$

where Ψ and Z are known functions. In the problems of this section the boundary conditions are of the form

$$
u|_{x=y(t)} = \Psi(y(t)); \quad y(t) = Z\left(\frac{\partial u}{\partial x}\right)\Big|_{x=y(t)}, \tag{4.3.24}
$$

or

$$
\begin{aligned}
&\frac{\partial u}{\partial x}\Big|_{x=y(t)} = \Psi(y(t));\\
&\frac{\partial u}{\partial t}\Big|_{x=y(t)} = Z\left(u, \frac{\partial u}{\partial x}, y\right)\Big|_{x=y(t)},
\end{aligned}
\tag{4.3.25}
$$

where Ψ and Z are known functions.

Thus we are dealing with problems in which the velocity of the moving boundary is determined not only by the desired u and its first derivative with respect to x, but also by the derivative of u with respect to t, i.e. by the derivative of highest order appearing in the equation. This difference is, of course, highly significant. The theory of such problems is not developed up to the present time.

Part Two

THE CLASSICAL STEFAN PROBLEM AND ITS GENERALIZATIONS

CHAPTER I

THE SINGLE-PHASE STEFAN PROBLEM WITH STRONG NONLINEARITY [136]

§1. Statement of the problem. Formulation of the basic results

We now investigate the following problem. Find $y(t)$ and $u(x,t)$ such that $u(x,t)$ and $\partial u(x,t)/\partial x$ are well defined and continuous in the closure of the parabolic domain

$$D = \{0 < x < y(\tau);\quad 0 < \tau \leqq t < T\}, \tag{1.1.1}$$

and $\partial^2 u/\partial x^2$ and $\partial u/\partial t$ are continuous in D, and satisfy the equation

$$L_a u + F\left(x,t,u,\frac{\partial u}{\partial x},y,\dot{y}\right) = 0,\quad (x,t) \in D \tag{1.1.2}$$

and the boundary conditions

$$\frac{\partial u}{\partial x} = f(t,u) \text{ for } x = 0;\quad t > 0; \tag{1.1.3}$$

$$u = \phi(x) \text{ for } t = 0;\quad 0 \leqq x \leqq y(0); \tag{1.1.4}$$

$$u = \psi(x) \text{ for } x = y(t);\quad t > 0; \tag{1.1.5}$$

$$\dot{y} = Z\left(t,u,\frac{\partial u}{\partial x},x\right) \text{ for } x = y(t);\quad t > 0;\quad y(0) = l > 0. \tag{1.1.6}$$

Here F, ϕ, ψ, f and Z are known functions.

We introduce the notation

$$u|_{x=0} = w;\quad \frac{\partial u}{\partial x} = q;\quad 0 < x < y(t);\quad q|_{x=y(t)} = v;\quad \dot{y}(t) = z. \tag{1.1.7}$$

Assume that f, ϕ, ψ, F and Z are defined for $0 \leqq x \leqq 1$, $0 \leqq t \leqq 1$, $0 \leqq y \leqq 1$ and all w, q, v and z, and in addition assume that:

a) F, f and Z are differentiable, ψ is twice differentiable and ϕ is thrice differentiable with respect to their arguments everywhere in the domain of definition.

b) If

$$\begin{aligned} &0 \leqq x \leqq 1; \quad 0 \leqq t \leqq 1; \quad 0 \leqq y \leqq 1; \quad |w| \leqq N_1; \\ &|u| \leqq N_2; \quad |v| \leqq N_3; \quad |q| \leqq N_4; \quad |z| \leqq N_5 \end{aligned} \tag{1.1.8}$$

then

$$\begin{aligned} &|f(t,w)| \leqq M_1; \quad |\psi(y)| \leqq M_2; \quad |\dot{\psi}(y)| \leqq M_2; \\ &|Z| \leqq M_3; \quad |F| \leqq M_4; \quad |\phi| \leqq M_0; \quad |\dot{\phi}| \leqq M_0 \end{aligned} \tag{1.1.9}$$

and

$$\begin{aligned} &\left|\frac{\partial f}{\partial t}\right| \leqq M_{1,1}; \quad \left|\frac{\partial f}{\partial w}\right| \leqq M_{1,2}; \quad |\ddot{\psi}| \leqq M_{2,1}; \\ &\left|\frac{\partial Z}{\partial t}\right| \leqq N_{5,1}; \quad \left|\frac{\partial Z}{\partial u}\right| \leqq N_{5,2}; \quad \left|\frac{\partial Z}{\partial q}\right| \leqq N_{5,3}; \quad \left|\frac{\partial Z}{\partial x}\right| \leqq N_{5,4}; \\ &\left|\frac{\partial F}{\partial x}\right| < N_{4,1}; \quad \left|\frac{\partial F}{\partial t}\right| \leqq N_{4,2}; \quad \left|\frac{\partial F}{\partial u}\right| \leqq N_{4,3}; \quad \left|\frac{\partial F}{\partial q}\right| \leqq N_{4,4}; \\ &\left|\frac{\partial F}{\partial y}\right| \leqq N_{4,5}; \quad \left|\frac{\partial F}{\partial z}\right| \leqq N_{4,6}; \quad |\ddot{\phi}| \leqq M_0; \quad |\dddot{\phi}| \leqq M_0; \end{aligned} \tag{1.1.10}$$

for M_1, $M_{i,j}$, $N_{i,j}$ positive constants.

c)

$$f(0,\phi(0)) = \dot{\phi}(0); \quad \phi(l) = \psi(l). \tag{1.1.11}$$

Under these conditions the following fundamental theorems are valid.

THEOREM 1. *If $u(x,t)$, $y(t)$ are solutions of problem* (1.1.2)—(1.1.6) *then u, w, q, v, y and z are solutions of the system of integral equations*

$$\begin{aligned} u = &- a^2 \int_0^t f(\tau,w)\, G(x,0,t-\tau)\, d\tau + \int_0^l \phi(\xi)\, G(x,\xi,t)\, d\xi \\ &+ \int_0^t d\tau \int_0^{y(\tau)} F(\xi,\tau,\cdots,z)\, G(x,\xi,t-\tau)\, d\xi \\ &+ \int_0^t [a^2 v(\tau) + \psi(y(\tau))\, z(\tau)]\, G(x,y(\tau),t-\tau)\, d\tau \\ &- a^2 \int_0^t \psi(y(\tau))\, \frac{\partial}{\partial \xi}\, G(x,y(\tau),t-\tau)\, d\tau \\ \equiv\ & U(t,x|u,w,q,v,y,z); \end{aligned} \tag{1.1.12$_1$}$$

$$w = U|_{x=0} = W(t|u,w,q,v,y,z); \tag{1.1.12$_2$}$$

$$\begin{aligned} q = {} & a^2 \int_0^t f(\tau,w)\frac{\partial}{\partial\xi}g(x,0,t-\tau)\,d\tau + \int_0^l \dot{\phi}(\xi)g(x,\xi,t)\,d\xi \\ & - \int_0^t d\tau \int_0^{y(\tau)} F(\xi,\tau,\cdots,z)\frac{\partial}{\partial\xi}g(x,\xi,t-\tau)\,d\xi \\ & - a^2\int_0^t v(\tau)\frac{\partial}{\partial\xi}g(x,y(\tau),t-\tau)\,d\tau \\ & + \int_0^t \dot{\psi}(y(\tau))z(\tau)g(x,y(\tau),t-\tau)\,d\tau \\ \equiv {} & Q(t,x|u,w,q,v,y,z); \end{aligned} \tag{1.1.12$_3$}$$

$$v = 2Q|_{x=y(t)} = V(t|u,w,q,v,y,z); \tag{1.1.12$_4$}$$

$$y(t) = l + \int_0^t z(\tau)\,d\tau = Y(t|z); \tag{1.1.12$_5$}$$

$$z(t) = Z(t,\psi(y),v,y). \tag{1.1.12$_6$}$$

Here g and G are Green's functions of the first and second boundary problems for the equation $L_a u = 0$ in the halfspace $x > 0$, i.e.

$$\begin{aligned} g(x,\xi,t) &= E(x-\xi,a^2t) - E(x+\xi,a^2t), \\ G(x,\xi,t) &= E(x-\xi,a^2t) + E(x+\xi,a^2t). \end{aligned} \tag{1.1.13}$$

THEOREM 2 (EQUIVALENCE THEOREM). *If $u,\cdots,z$ are solutions of the system* (1.1.12$_i$) ($i = 1,\cdots,6$), *satisfying a Lipschitz condition of the form*

$$|f(t) - f(\tau)| < A(\sqrt{t}+\sqrt{\tau})^{-1}|t-\tau|,$$

with respect to each of their arguments, then u and y are solutions of the problem (1.1.2)—(1.1.6).

THEOREM 3 (EXISTENCE AND UNIQUENESS). *The unique solution of the system* (1.1.12$_i$) ($i = 1,\cdots,6$) *satisfying the conditions of Theorem* 2 *can be constructed by the method of Picard approximations, if we begin the iteration process with any functions u_0, w_0, q_0, v_0, y_0, z_0 having bounded partial derivatives with respect to each of their arguments; the solution satisfies the conditions*

$$\dot{\phi}(0) = f(0,w_0(0)); \quad \dot{\phi}(l) = q_0(l,0) = v_0(0); \quad \dot{y}_0(t) = z_0(t). \tag{1.1.14}$$

The iteration process is defined by the scheme

$$w_n = W_n; \quad u_n = U_n; \quad v_n = V_n; \quad q_n = Q_n; \quad z_n = Z_n; \quad y_n = Y_n, \tag{1.1.15}$$

where the functions

$$\begin{aligned} W_n &= W(t|w_{n-1}, u_{n-1}, q_{n-1}, v_{n-1}, y_{n-1}, z_{n-1}) \\ U_n &= U(t, x|w_n, u_{n-1}, q_{n-1}, v_{n-1}, y_{n-1}, z_{n-1}) \\ V_n &= V(t|w_n, u_n, q_{n-1}, v_{n-1}, y_{n-1}, z_{n-1}) \\ Q_n &= Q(t, x|w_n, u_n, q_{n-1}, v_n, y_{n-1}, z_{n-1}) \\ Z_n &= Z(t, \psi(y_{n-1}), v_n, y_{n-1}) \\ Y_n &= Y(t|z_n) \end{aligned} \tag{1.1.16}$$

converge in the time interval $(0, T)$, with $T > 0$ depending on bounds for f, F, Z, ψ and ϕ, on bounds for every partial derivative of $f, \cdots, \phi$ arising in the conditions of the problem, and on $l = y(0)$, where $T \to 0$ as $l \to 0$.

The solution $q(x,t)$, $w(t)$, $v(t)$ of the system satisfies Lipschitz conditions of the form

$$\begin{aligned} &|w(t) - w(\tau)| < \frac{A|t-\tau|}{\sqrt{t}+\sqrt{\tau}} \equiv A^*(t,\tau) \\ &|v(t) - v(\tau)| < A^*(t,\tau) \\ &|q(x,t) - q(x,\tau)| < A^*(t,\tau) \end{aligned} \tag{1.1.17}$$

with the constant A defined uniformly for all x.

THEOREM 4 (STABILITY). *The solution of the system* (1.1.12$_i$) ($i = 1, \cdots, 6$) *is stable relative to small variations of all of the "data" of the problem.*

The precise formulation of Theorem 4 is given below in §5 of this chapter.

§2. Reduction to an integral equation. Theorem of equivalence

A. Our goal now is to prove Theorems 1 and 2. With this in mind, we begin by investigating the following subsidiary problem.

Let $X_i(t)$ ($i = 1, 2$) have continuous derivatives of the first order for $t > 0$; in addition suppose that $\sqrt{t}\, X_i(t)$ is continuous for $t \geqq 0$ and

$$X_1(t) < x < X_2(t); \quad t > 0.$$

We consider the parabolic domain

$$D = \{X_1(t) < x < X_2(t); \quad t > 0\}$$

with boundary

$$\Gamma = \{x = X_i(\tau);\ 0 \leqq \tau \leqq t;\ X_1(0) \leqq x \leqq X_2(0);\ t = 0.\}$$

Suppose now that u is uniformly bounded in D, continuous in $\overline{D} = D \cup \Gamma$ together with $\partial u/\partial x$ everywhere except possibly at the point $(X_i(0), 0)$ $(i = 1, 2)$, and within D satisfies the equation

$$L_a u + F(x,t) = 0. \tag{1.2.1}$$

Then u admits the integral representation [**50**]

$$\begin{aligned} u(x,t) = &\int_\Gamma \left[a^2 \frac{\partial u}{\partial \xi} E(x-\xi, a^2(t-\tau)) - u(\xi,\tau) \frac{\partial}{\partial \xi} E(x-\xi, a^2(t-\tau)) \right] d\tau \\ &+ \int_\Gamma u(\xi,\tau) E(x-\xi, a^2(t-\tau))\, d\xi \\ &+ \int_D F(\xi,\tau) E(x-\xi, a^2(t-\tau))\, d\xi d\tau. \end{aligned} \tag{1.2.2}$$

The equation (1.2.2) will be called the fundamental formula. In it one may substitute for E the Green's function of any of the three classical boundary problems of the equation $L_a u = 0$ defined for the domain D or for an arbitrary domain D' covering D.

Suppose that D lies on one side of the line $x = x_0$. We then replace E in (1.2.2) by the Green's function of the second boundary problem for the equation $L_a u = 0$ on the ray $x > x_0$ $(x < x_0)$:

$$G(x,\xi,t) = E(x-\xi, a^2 t) + E(x+\xi-2x_0, a^2 t) \tag{1.2.3}$$

and introduce the notation

$$u|_{x=X_i(t)} = w_i(t);\quad \frac{\partial}{\partial x} u \Big|_{x=X_i(t)} = v_i(t),\quad u|_{t=0} = \phi(x). \tag{1.2.4}$$

In view of the differentiability of $X_j(t)$ $(j = 1, 2)$, equation (1.2.2) can be written in the form

$$\begin{aligned} u(x,t) = &\int_0^t [a^2 v_i(\tau) + w_i(\tau)\dot{X}_i(\tau)] G(x, X_i(\tau), t-\tau)\,|_{i=1}^2 d\tau \\ &+ \int_{X_1(0)}^{X_2(0)} \phi(\xi) G(x,\xi,t)\, d\xi - a^2 \int_0^t w_i(\tau) \frac{\partial}{\partial \xi} G(x, X_i(\tau), t-\tau)\,|_{i=1}^2 d\tau \\ &+ \int_0^t d\tau \int_{X_1(\tau)}^{X_2(\tau)} F(\xi,\tau) G(x,\xi,t-\tau)\, d\xi. \end{aligned} \tag{1.2.5}$$

We now set

$$q(x,t) = \frac{\partial}{\partial x} u(x,t); \quad X_1(t) < x < X_2(t); \quad t > 0 \tag{1.2.6}$$

and find its integral representation.

Let $g(x,\xi,t)$ be the Green's function of the first boundary problem for the halfline $x > x_0$:

$$g(x,\xi,t) = E(x-\xi, a^2t) - E(x+\xi-2x_0, a^2t). \tag{1.2.7}$$

It is obvious that g and G are mutually conjugate. Let $X_1(t) < x < X_2(t)$, $t > 0$. To calculate q we differentiate within the integral sign; using the conjugacy of G and g, together with the validity, when necessary, of integration by parts, we find that

$$\begin{aligned} q(x,t) = &\sum_{j=1}^{2} (-1)^j [w_j(0) - \phi(X_j(0))] g(x, X_j(0), t) \\ &+ \int_{X_1(0)}^{X_2(0)} \dot{\phi}(\xi) g(x,\xi,t)\, d\xi \\ &+ \sum_{j=1}^{2} (-1)^{j+1} \left\{ - \int_0^t \dot{w}_j(\tau) g(x, X_j(\tau), t-\tau)\, d\tau \right. \\ &\qquad \left. + a^2 \int_0^t v_j(\tau) \frac{\partial}{\partial \xi} g(x, X_j(\tau), t-\tau)\, d\tau \right\} \\ &- \int_0^t d\tau \int_{X_1(\tau)}^{X_2(\tau)} F(\xi,\tau) \frac{\partial}{\partial \xi} g(x,\xi,t-\tau)\, d\xi. \end{aligned} \tag{1.2.8}$$

Now, going to the limit for $x \to X_k(t) + (-1)^{k+1} \cdot 0$ $(k = 1, 2)$ in (1.2.8), and using (1.2.4) and the theorem on discontinuity of the heat potential of a double layer, according to which[1)]

$$\begin{aligned} &\lim_{x \to X_j(t) + (-1)^{j+1} \cdot 0} (-1)^{j+1} a^2 \int_0^t v_j(\tau) \frac{\partial}{\partial \xi} g(x, X_j(\tau), t-\tau)\, d\tau \\ &= \frac{1}{2} v_j(t) + (-1)^{j+1} a^2 \int_0^t v_j(\tau) \frac{\partial}{\partial \xi} g(X_j(t), X_j(\tau), t-\tau)\, d\tau \end{aligned} \tag{1.2.9}$$

we find that

[1)] (1.2.9) is automatically valid if $X_j(t)$ satisfies a Hölder condition and $\sqrt{t}\, v_j(t)$ is continuous for $t \geq 0$ (see [50]).

$$
\begin{aligned}
v_k(t) = 2 \Bigg\{ & \sum_{j=1}^{2} (-1)^j [w_j(0) - \phi(\mathrm{X}_j(0))] g(\mathrm{X}_k(t), \mathrm{X}_j(0), t) \\
& + \int_{\mathrm{X}_1(0)}^{\mathrm{X}_2(0)} \dot{\phi}(\xi) g(x, \xi, t)\, d\xi \\
& + \sum_{j=1}^{2} (-1)^{j+1} \Bigg[- \int_0^t \dot{w}_j(\tau) g(\mathrm{X}_k(t), \mathrm{X}_j(\tau), t-\tau)\, d\tau \\
& \qquad + a^2 \int_0^t v_j(\tau) \frac{\partial}{\partial \xi} g(\mathrm{X}_k(t), \mathrm{X}_j(\tau), t-\tau)\, d\tau \Bigg] \\
& \qquad - \int_0^t d\tau \int_{\mathrm{X}_1(\tau)}^{\mathrm{X}_2(\tau)} F(\xi, \tau) \frac{\partial}{\partial \xi} g(\mathrm{X}_k(t), \xi, t-\tau)\, d\tau. \Bigg\}
\end{aligned}
\tag{1.2.10}
$$

Analogously, passing to the limit for $x \to \mathrm{X}_k(t) + (-1)^{k+1} \cdot 0$ in (1.2.5), we obtain

$$
\begin{aligned}
w_k(t) = 2 \Bigg\{ & \sum_{j=1}^{2} (-1)^j \int_0^t \Big([a^2 v_j(\tau) + w_j(\tau) \dot{\mathrm{X}}_j(\tau)] G(\mathrm{X}_k(t), \mathrm{X}_j(\tau), t-\tau) \\
& \qquad - a^2 w_j(\tau) \frac{\partial}{\partial \xi} G(\mathrm{X}_k(t), \mathrm{X}_j(\tau), t-\tau) \Big)\, d\tau \\
& + \int_{\mathrm{X}_1(0)}^{\mathrm{X}_2(0)} \phi(\xi) G(\mathrm{X}_k(t), \xi, t)\, d\xi \\
& \qquad + \int_0^t d\tau \int_{\mathrm{X}_1(\tau)}^{\mathrm{X}_2(\tau)} F(\xi, \tau) G(\mathrm{X}_k(t), \xi, t-\tau)\, d\xi \Bigg\}.
\end{aligned}
\tag{1.2.11}
$$

We note that if the boundary and initial conditions of the problem are consistent with respect to continuity, then the term outside the integral in (1.2.10) and (1.2.8) vanishes.

In (1.2.5)—(1.2.11) the terms u and $\partial u / \partial x$ appear simultaneously at the moving boundary curves $x = \mathrm{X}_j(t)$. On each of the boundary curves we may independently assign one of these variables or a combination of both of them. For definiteness we assume below that these are given by $v_1(t)$ and $w_2(t)$, respectively. Thus we take $k = 2$ in (1.2.10) and $k = 1$ in (1.2.11).

B. Using equations (1.2.10) and (1.2.11), we can determine a function $u(x, t)$ satisfying (1.2.1) under the conditions

$$
u|_{x=\mathrm{X}_2(t)} = w_2(t); \quad \frac{\partial}{\partial x} u \Big|_{x=\mathrm{X}_1(t)} = v_1(t); \quad u|_{t=0} = \phi(x).
\tag{1.2.12}
$$

For the sake of simplicity we assume that $\sqrt{t}\dot{X}_i(t)$ $(i = 1, 2)$ is continuous for $t \geqq 0$, and that

$$X_1(0) < x < X_2(0); \tag{1.2.13}$$

in addition we assume continuity of the boundary and initial data,

$$v_1(0) = \dot{\phi}(X_1(0)); \quad w_2(0) = \phi(X_2(0)). \tag{1.2.14}$$

Suppose that v_2 and w_1 are found from (1.2.10) and (1.2.11).[2] If we introduce them into (1.2.5) we can show that u actually satisfies all of the conditions of problem (1.2.1), (1.2.12), if only $F(x,t)$ is smooth enough so that the heat potential

$$\int_0^t d\tau \int_D F(\xi,\tau)\, E(x-\xi, a^2(t-\tau))\, d\xi$$

is a solution to equation (1.2.1).[3]

Under our assumptions it is obvious that $u(x,t)$ satisfies equation (1.2.1) and the initial conditions (1.2.12). By virtue of the method of constructing equation (1.2.10) the condition

$$\lim_{x \to X_1(t)+0} u = w_1(t) \tag{1.2.15}$$

is obvious. Furthermore, passing to the limit for $t \to 0$ and using the equality[4]

$$\lim_{t \to +0} \int_\alpha^\beta \phi(\xi)\, E(x-\xi, a^2 t)\, d\xi = \begin{cases} 0; & x \notin [\alpha, \beta] \\ \dfrac{1}{2}\phi(x); & x = \alpha \text{ or } x = \beta \\ \phi(x); & x \in (\alpha, \beta), \end{cases} \tag{1.2.16}$$

which is valid for any continuous function, we find that

$$\lim_{t \to 0} w_1(t) = \phi(X_1(0)). \tag{1.2.17}$$

We now go to the limit in (1.2.5) for $x \to X_2(t)$. Using (1.2.9) we find that

---

[2] Under our assumptions the system (1.2.10), (1.2.11) is an ordinary system of equations of Volterra type with a polar kernel. Consequently it has a unique solution in the class of continuous functions.

[3] This is automatically true if F satisfies a Hölder condition with respect to x and t with a uniformly defined constant [50].

[4] We note that if $\alpha = X_1(0)$ and $\beta = X_2(0)$ then (1.2.16) is valid for $x = X_j(t)$ $(j = 1, 2)$ or for any continuous $x = x(t) \in (\alpha, \beta)$.

$$\lim_{x \to X_2(t) - 0} u = \frac{1}{2} w_2(t) + U(X_2(t), t), \tag{1.2.18}$$

where U denotes the right side of (1.2.5). This shows that

$$\lim_{t \to 0} \lim_{x \to X_2(t)} u = \frac{1}{2} w_2(0) + \frac{1}{2} \phi(X_2(0)). \tag{1.2.19}$$

Hence since $w_2(0) = \phi(X_2(0))$ we conclude that u is continuous at the point $(X_2(0), 0)$. A comparison of (1.2.12), (1.2.15) and (1.2.19) shows that u is continuous in D.

However, in this case, from the method of constructing (1.2.8) it is obvious that the right side of (1.2.8) is actually equal to $\partial u / \partial x$. This shows that

$$\lim_{x \to X_2(t)} \frac{\partial u}{\partial x} = \frac{1}{2} v_2(t) + Q(X_2(t), t), \tag{1.2.20}$$

where Q represents the right side of (1.2.8). Comparing (1.2.20) and (1.2.10) (for $k = 2$) shows us that in reality

$$\lim_{x \to X_2(t)} \frac{\partial u}{\partial x} = v_2(t). \tag{1.2.21}$$

We note now that by the assumptions of the problem the equation

$$v_1(0) = \dot{\phi}(X_1(0)). \tag{1.2.22}$$

which has not been used until now, is valid. Passing to the limit for $x \to X_1(t) + 0$ in (1.2.8), we find that

$$\lim_{x \to X_1(t)} \frac{\partial u}{\partial x} = \frac{1}{2} v_1(t) + Q(X_1(t), t). \tag{1.2.23}$$

Hence in the limit for $t \to 0$ it follows that

$$\lim_{t \to 0} \lim_{x \to X_1(t)} \frac{\partial u}{\partial x} = \frac{1}{2} v_1(0) + \frac{1}{2} \dot{\phi}(X_1(0)), \tag{1.2.24}$$

which by virtue of (1.2.23) shows that

$$\lim_{t \to 0} \lim_{x \to X_1(t)} \frac{\partial u}{\partial x} = \dot{\phi}(X_1(0)). \tag{1.2.25}$$

Analogously, passing to the limit for $t \to 0$ in (1.2.10) gives us

$$\lim_{t \to 0} \lim_{x \to X_2(t) - 0} \frac{\partial u}{\partial x} = \dot{\phi}(X_2(0)). \tag{1.2.26}$$

Thus it remains for us only to prove the validity of the first two conditions of (1.2.12). With this aim in mind, we note that the continuity of u and $\partial u/\partial x$ imply the validity of the fundamental formula (1.2.2) for u. Let

$$\lim_{x\to X_2(t)-0} u = w_2^*(t); \quad \lim_{x\to X_1(t)+0} \frac{\partial u}{\partial x} = v_1^*(t). \tag{1.2.27}$$

In this case (1.2.2) (with E replaced by G) yields, as in (1.2.5)

$$\begin{aligned} u(x,t) = &\int_0^t [a^2 v_2(\tau) + w_2^*(\tau)\dot{X}_2(\tau)]G(x, X_2(\tau), t-\tau)\,d\tau \\ &- \int_0^t [a^2 v_1^*(\tau) + w_1(\tau)\dot{X}_1(\tau)]G(x, X_1(\tau), t-\tau)\,d\tau \\ &- a^2\int_0^t w_2^*(\tau)\frac{\partial}{\partial\xi}G(x, X_2(\tau), t-\tau)\,d\tau \\ &+ a^2\int_0^t w_1(\tau)\frac{\partial}{\partial\xi}G(x, X_1(\tau), t-\tau)\,d\tau \\ &+ \int_{X_1(0)}^{X_2(0)} \phi(\xi)G(x,\xi,t)\,d\xi \\ &+ \int_0^t d\tau \int_{X_1(\tau)}^{X_2(\tau)} F(\xi,\tau)G(x,\xi,t-\tau)\,d\xi. \end{aligned} \tag{1.2.28}$$

Assume

$$v(t) = v_1(t) - v_1^*(t); \quad w(t) = w_2(t) - w_2^*(t). \tag{1.2.29}$$

Subtracting (1.2.28) from (1.2.5), we obtain

$$\begin{aligned} 0 = \int_0^t \Big\{ w(\tau)\left[\dot{X}_2(\tau) - a^2\frac{\partial}{\partial\xi}\right]G(x, X_2(\tau), t-\tau) \\ - a^2 v(\tau)G(x, X_1(\tau), t-\tau)\Big\}\,d\tau. \end{aligned} \tag{1.2.30}$$

Now, passing to the limit for $x \to X_2(t) - 0$, we find with the help of (1.2.9) that

$$\begin{aligned} w(t) = 2\int_0^t \Big\{ a^2 v(\tau)G(X_2(t), X_1(\tau), t-\tau) \\ + w(\tau)\left[a^2\frac{\partial}{\partial\xi} - \dot{X}_2(\tau)\right]G(X_2(t), X_2(\tau), t-\tau)\Big\}\,d\tau. \end{aligned} \tag{1.2.31}$$

Differentiating (1.2.30) with respect to x and passing to the limit for

$x \to X_1(t) + 0$ we analogously obtain

$$v(t) = -2\int_0^t \left[a^2 v(\tau) \frac{\partial}{\partial \xi} g(X_1(t), X_1(\tau), t-\tau) + w(\tau) \frac{d}{d\tau} g(X_1(t), X_2(\tau), t-\tau) \right] d\tau. \tag{1.2.32}$$

By assumption $X_2 - X_1$ does not vanish. Therefore the expression in square brackets in the second integral on the right in (1.2.32) remains bounded. Consequently the system (1.2.31), (1.2.32) is a homogeneous system of integral equations of Volterra type of second order with polar kernel. From the general theory of such equations we know that the solution is unique. Thus $v \equiv w \equiv 0$ and the assertion is proved.

REMARK. The proof must yet be given in an unbounded domain. In this case one must demand that $\phi_i(x)$ not grow to infinity faster than $\exp(cx^2)$, where $c > 0$, and that $t > 0$ be sufficiently small.

C. We now return to the original Stefan problem (1.1.2)—(1.1.6). We assume that its solution exists and is defined for $0 \leqq t \leqq T$, and, in addition, that for such t

$$0 < \rho \leqq y(t). \tag{1.2.33}$$

With these assumptions $F(x,t,u,\partial u/\partial x, y, \dot{y})$, $f(t,u)$, ψ and $Z(t,u,\partial u/\partial x, y)$ can be assumed to be known functions of t and x. Consequently we can apply the theory developed above to u and y. Moreover,

$$X_1(t) \equiv 0; \quad X_2(t) \equiv y(t); \quad x_0 = 0. \tag{1.2.34}$$

With such a choice of x_0 the terms of (1.2.5) depending on $u(0,t)$ drop out. Now, comparing (1.2.5), (1.2.8), (1.2.10) and (1.2.11) with equations $(1.1.12_1)$—$(1.1.12_4)$, and condition (1.2.14) with (1.1.11), we see that Theorem 1 is valid.

Conversely, if a solution of the system $(1.1.12_1)$—$(1.1.12_4)$ exists, satisfying a Lipschitz condition with respect to all of its arguments, then when considered as a function of t and x, F will have enough smoothness to use it for the proof of an equivalence theorem for the subsidiary problem. But this shows the correctness of Theorem 2, which is thus proved.

§3. Existence of the solution in the small

A. We now turn to the proof of the existence theorem. Let $w_0(t)$, $u_0(x,t)$, $q_0(x,t)$, $v_0(t)$, $z_0(t)$ and $y_0(t)$ be arbitrary differentiable functions which for $0 \leqq x \leqq 1$, $0 \leqq t \leqq 1$ satisfy the inequalities (1.1.8) and

$$
\begin{gathered}
|\sqrt{t}\dot{w}_0(t)| < L_1; \quad \left|\sqrt{t}\frac{\partial u_0}{\partial t}\right| < L_2; \quad |\sqrt{t}\dot{v}_0(t)| < L_3; \\
\left|\sqrt{t}\frac{\partial q_0}{\partial t}\right| < L_4; \quad |\sqrt{t}\dot{z}_0(t)| < L_5; \quad \left|\frac{\partial q_0}{\partial x}\right| < L^*,
\end{gathered} \tag{1.3.1}
$$

where $L_1, \cdots, L_5, L^*$ are certain fixed constants, in addition to the conditions

$$
\begin{gathered}
\dot{y}_0(t) = z_0(t); \quad q_0(y_0(t), t) = v_0(t); \quad u_0(0, t) = w_0(t); \\
u_0(x, 0) = \phi(x); \quad q_0(x, 0) = \dot{\phi}(x); \quad w_0(0) = \phi(0); \quad y_0(0) = l.
\end{gathered} \tag{1.3.2}
$$

We now define $w_n, \cdots, y_n$ by equation (1.1.15). Our immediate goal is to prove the existence of some value $T > 0$ such that for $0 \leqq x \leqq 1$, $0 \leqq t \leqq T$ the set of functions $\{w_n(t)\}$, $\{z_n(t)\}$, $\{u_n(x,t)\}$, $\{q_n(x,t)\}$, $\{v_n(t)\}$ is uniformly bounded and equicontinuous.[5] Hence from the theorem of Arzelà and the uniform continuity of the operators $W, \cdots, Z$ appearing on the right sides of equations $(1.1.12_1)$—$(1.1.12_6)$ we will be able to conclude that in the small a solution to the problem for the system of integral equations exists.

We begin by proving the existence of a number $T_0 > 0$ such that if w, u, q, v and z satisfy conditions (1.1.8), (1.1.14) for $0 \leqq x \leqq 1$, $0 \leqq t \leqq T_0$, then at these points the same conditions will be satisfied by the values of the operators $W, \cdots, Z$.

We make use of the notation (0.2.14) (see §2 of the Introduction). In this notation

---

[5] Within the expressions for the functionals U, Q, $\partial U/\partial t$, $\partial Q/\partial t$, $\partial Q/\partial x$ there is a double layer potential of the form

$$
a^2 \int_0^t \mu(\tau) \frac{\partial}{\partial \xi} E(x - y(\tau), a^2(t - \tau))\, d\tau,
$$

which undergoes a discontinuity as one tends to the point $(y(t), t)$. For the construction of the solution to the system $(1.1.12_i)$ $(i = 1, \cdots, 6)$ we assume that $0 < x < y(t)$. In the definition of the iteration process it is therefore proper to assume that u_n and q_n are defined for $0 < x < y_n$, i.e. generally speaking in a domain which is wider than the domain of continuity of the functionals $U, \cdots, \partial Q/\partial x$. In order to avoid this difficulty in our iteration process we will assume that the functionals $U, \cdots, \partial Q/\partial x$ are extended by continuity. In other words, for the needs of our iteration scheme we will add to the expressions appearing on the right sides of the equations defining $U, \cdots, \partial Q/\partial x$ expressions of the form $\mu(t)\eta[x - y(t)]$, where η is a unit function, i.e.

$$
\eta(x) = \begin{cases} 0; & x < 0 \\ \frac{1}{2}; & x = 0 \\ 1; & x > 0 \end{cases} \tag{1.3.3}
$$

$$
\begin{aligned}
&U(x,t|f,\phi,F,u,q,w,v,y,z)\\
&\quad=\sum_{i=1}^{2}[I_{1,i}(x,t,0,\tau,a|-f(\tau,w(\tau)))\\
&\quad\quad+I_{1,i}(x,t,y(\tau),\tau,a|v(\tau)+a^{-2}\psi(y(\tau))z(\tau))\\
&\quad\quad+I_{2,i}(x,t,y(\tau),\tau,a|-\psi(y(\tau)))+I_{3,i}(x,t,\xi,0,l,a|\phi(\xi))\\
&\quad\quad+I_{4,i}(x,t,\xi,\tau,0,y(\tau),a|F(\xi,\tau,u,q,y,z))],
\end{aligned}
\tag{1.3.4}
$$

$$
\begin{aligned}
&Q(x,t|f,\dot\phi,\dot\psi,F,u,q,w,v,y,z)\\
&\quad=\sum_{i=1}^{2}(-1)^{i+1}[I_{1,i}(x,t,y(\tau),\tau,a|a^{-2}\dot\psi(y(\tau))z(\tau))\\
&\quad\quad+I_{2,i}(x,t,0,\tau,a|f(\tau,w(\tau)))+I_{2,i}(x,t,y(\tau),\tau,a|-v(\tau))\\
&\quad\quad+I_{3,i}(x,t,\xi,0,l,a|\dot\phi(\xi))\\
&\quad\quad+I_{5,i}(x,t,\xi,\tau,0,y(\tau),a|-F(\xi,\tau,u,q,y,z))].
\end{aligned}
\tag{1.3.5}
$$

The values of the functions $I_{i,j}$ $(i=1,\cdots,5;\ j=1,2)$ are estimated by means of the inequalities (1.1.8). Let $T_0^*>0$ be so small that

$$0<\rho\leqq l-N_5T_0^*<l+N_5T_0^*\leqq 1, \tag{1.3.6}$$

where ρ is some constant. Then automatically

$$0<\rho<y=l+\int_0^t z(\tau)\,d\tau\leqq 1;\quad 0\leqq t\leqq T_0^*. \tag{1.3.7}$$

Let $x(t)$ and $\xi(\tau)$ be arbitrary differentiable functions such that

$$0\leqq\xi(\tau)\leqq 1;\quad 0\leqq x(t)\leqq 1;\quad |\dot x(t)|\leqq N_5;\quad |\dot\xi(\tau)|\leqq N_5, \tag{1.3.8}$$

for $0\leqq\rho\leqq t\leqq T_0^*$ and suppose that the fixed argument function f in the definition (0.2.14) satisfies the inequality

$$|f|<M \text{ for } 0\leqq x;\quad \xi\leqq 1;\quad 0\leqq\tau\leqq t\leqq T_0^*. \tag{1.3.9}$$

It is then obvious that

$$|I_{3,i}(x(t),t,\xi,\alpha,\beta,a|f)|\leqq\frac{M}{\sqrt{\pi}}\int_c^a\exp(-\lambda^2)\,d\lambda\leqq M; \tag{1.3.10}$$

$$|I_{1,i}(x(t),t,\xi(\tau),\tau,a|f)|\leqq\frac{aM}{2\sqrt{\pi}}\int_0^t\frac{d\tau}{\sqrt{t-\tau}}=\frac{aM}{\sqrt{\pi}}\sqrt{t}; \tag{1.3.11}$$

$$|I_{4,i}(x(t),t,\xi,\tau,\alpha,\beta,a|f)|\leqq\frac{M}{\sqrt{\pi}}\int_0^t d\tau\left|\int_c^d\exp(-\lambda^2)\,d\lambda\right|<Mt; \tag{1.3.12}$$

$$|I_{5,i}(x(t),t,\xi,\tau,\alpha,\beta,a|f)| \leqq \frac{M}{a\sqrt{\pi}}\int_0^t \frac{d\tau}{\sqrt{t-\tau}}\left|\int_c^d \lambda\exp(-\lambda^2)\,d\lambda\right| < \frac{M\sqrt{t}}{a\sqrt{\pi}}. \tag{1.3.13}$$

Here

$$c=\frac{x(t)+(-1)^i\alpha}{2a\sqrt{t}};\quad d=\frac{x(t)+(-1)^i\beta}{2a\sqrt{t}}. \tag{1.3.14}$$

Moreover,

$$\begin{aligned} I_{2,1}&(x(t),t,\xi(\tau),\tau,a|f)\\ &=\frac{1}{4a\sqrt{\pi}}\int_0^t f\cdot\frac{x(t)-\xi(\tau)}{(t-\tau)^{3/2}}\cdot\exp\left[-\frac{(x(t)-\xi(\tau))^2}{4a^2(t-\tau)}\right]d\tau\\ &=\frac{1}{4a\sqrt{\pi}}\int_0^t f\cdot\frac{x(t)-\xi(t)}{(t-\tau)^{3/2}}\cdot\exp\left[-\frac{(x(t)-\xi(t))^2}{4a^2(t-\tau)}\right]\\ &\quad\cdot\exp\left[-\frac{(\xi(t)-\xi(\tau))^2}{4a^2(t-\tau)}-(x(t)-\xi(t))\frac{\xi(t)-\xi(\tau)}{2a^2(t-\tau)}\right]\cdot d\tau\\ &\quad+\frac{1}{4a\sqrt{\pi}}\int_0^t f\cdot\frac{\xi(t)-\xi(\tau)}{(t-\tau)^{3/2}}\cdot\exp\left[-\frac{(x(t)-\xi(\tau))^2}{4a^2(t-\tau)}\right]d\tau\\ &=I'_{2,1}+I''_{2,1}. \end{aligned} \tag{1.3.15}$$

Hence from the inequalities $|\dot{\xi}(\tau)|<N_5$; $\xi(t)\in[0,1]$; $x(t)\in[0,1]$ it follows immediately that

$$|I_{2,1}(x(t),t,\xi(\tau),\tau,a|f)|<M\left[\epsilon A\frac{2}{\sqrt{\pi}}\int_{|x(t)-\xi(t)|/2a\sqrt{t}}^{\infty}\exp(-\lambda^2)\,d\lambda+N_5\cdot\frac{1}{4a\sqrt{\pi}}\int_0^t\frac{d\tau}{\sqrt{t-\tau}}\right], \tag{1.3.16}$$

where

$$A=\frac{1}{2}\exp\left(\frac{N_5}{2a^2}\right);\quad \epsilon=\begin{cases}1 & \text{for } x(t)\neq\xi(t)\\ 0 & \text{for } x(t)=\xi(t).\end{cases} \tag{1.3.17}$$

Thus

$$|I_{2,1}(x(t),t,\xi(\tau),\tau|f)|<M\left[\epsilon A\operatorname{erfc}\frac{|x(t)-\xi(t)|}{2a\sqrt{t}}+\frac{N_5}{2a\sqrt{\pi}}\sqrt{t}\right]. \tag{1.3.18}$$

Finally, to estimate $I_{2,2}$ we use the result obtained for $I_{2,1}$, replacing the quantity ξ by the quantity $-\xi$. Thus

$$(1.3.19)\quad |I_{2,2}(x(t),t,\xi(\tau),\tau|f)| < M\left[A\,\mathrm{erfc}\frac{|x(t)+\xi(t)|}{2a\sqrt{t}} + \frac{N_5}{2a\sqrt{\pi}}\sqrt{t}\right].$$

Comparing the estimates (1.3.10)—(1.3.13), (1.3.18) and (1.3.19) with (1.3.4) and (1.3.5) and taking (1.1.8) and (1.1.9) into account, we obtain

$$(1.3.20)\quad \begin{aligned} |U| < \Big\{&\frac{2\sqrt{t}}{\sqrt{\pi}}\left[a(M_1+N_3)+\frac{3}{2a}M_2N_5\right]+2M_0+2M_4t \\ &+AM_2\left(\epsilon\,\mathrm{erfc}\frac{|x-y(t)|}{2a\sqrt{t}}+\mathrm{erfc}\frac{|x+y(t)|}{2a\sqrt{t}}\right)\Big\}; \end{aligned}$$

$$(1.3.21)\quad \begin{aligned} |Q| < \Big\{&\frac{2\sqrt{t}}{\sqrt{\pi}}\left[2M_4+N_5(M_1+M_2+M_3)\right]+M_1A(1+\epsilon)\,\mathrm{erfc}\frac{x}{2a\sqrt{t}} \\ &+2M_0+N_3A\left(\epsilon\,\mathrm{erfc}\frac{|x-y(t)|}{2a\sqrt{t}}+\mathrm{erfc}\frac{|x+y(t)|}{2a\sqrt{t}}\right) \\ &+N_3\eta(x-y(t))\Big\}. \end{aligned}$$

Here $\epsilon \geqq 0$ is defined in agreement with (1.3.17).

We now require the inequality condition

$$(1.3.22_1)\qquad N_3 > 3M_0$$

with fixed N_3. In addition we require

$$(1.3.22_2)\qquad N_5 > M_3$$

with fixed N_5. This is possible, since M_0 is not dependent upon u, w, q, v and z, while M_3 depends only upon M_2 and is independent of u, w, q, v, z and N_3, since

$$(1.3.22_3)\qquad Z = Z(t,\psi(y),v,y) \qquad (M_3 \geqq |Z|).$$

We will also assume the validity of the inequality

$$(1.3.22_4)\qquad N_1;\quad N_2 > 3(M_0+AM_2);\quad N_4 > 3(M_0+AM_1)+AN_3+N_3$$

and we fix the values of N_1, N_2 and N_4. This can be done for fixed N_3 and N_5, since M_0 and M_2 are not dependent on N_i $(i=1,\cdots,5)$, A depends only on N_5, and M_1 depends only on N_1.

After fixing N_i $(i=1,\cdots,5)$ it is obviously possible to choose $T_0 > 0$ such that for $0 \leqq t \leqq T_0 \leqq T_0^*$ we have simultaneously

$$(1.3.23)\qquad |W| < N_1;\quad |U| < N_2;\quad |V| < N_3;\quad |Q| < N_4;\quad |Z| < N_5,$$

which completes the proof of our assertion.

B. Suppose now that u, w, q, v and z satisfy not only (1.1.8) and

(1.1.14) but also (1.3.1); we shall prove that in this case the values of the functionals W, U, Q, V and Z satisfy (1.3.1) if $0 \leqq x \leqq 1$, $0 \leqq \xi \leqq 1$, $0 \leqq \tau \leqq t \leqq T \leqq T_0$, where $T > 0$ is sufficiently small. We will not indicate every detail of the evaluation of $dW/dt, \cdots, dZ/dt$, but merely indicate the general steps.

1. In every integral with respect to τ we replace τ by $(t - \tau)$ and revert to the former variable in the final calculations.

2. In order to differentiate the Green's function with respect to t we use the relations

$$\frac{d}{dt} G(x(t), \xi(t-\tau), \tau)$$

$$= \dot{\xi}(t-\tau) \frac{\partial}{\partial \xi} G(x(t), \xi(t-\tau), \tau) - \dot{x}(t) \frac{\partial}{\partial \xi} g(x(t), \xi(t-\tau), \tau);$$

$$a^2 \frac{d}{dt} \frac{\partial}{\partial \xi} G(x(t), \xi(t-\tau), \tau)$$

(1.3.24)

$$= \dot{\xi}(t-\tau) \left[\frac{d}{d\tau} + \dot{\xi}(t-\tau) \frac{\partial}{\partial \xi} \right] G(x(t), \xi(t-\tau), \tau)$$

$$- \dot{x}(t) \left[\frac{d}{d\tau} + \dot{\xi}(t-\tau) \frac{\partial}{\partial \xi} \right] g(x(t), \xi(t-\tau), \tau)$$

as well as analogous expressions for dg/dt and $(d/dt)\,\partial g/\partial \xi$.

3. We integrate by parts every integral containing $dG/d\tau$ and $dg/d\tau$. We thus obtain

$$\frac{dU}{dt} = -a^2 \int_0^t \frac{d}{d\tau} f(\tau, w(\tau)) G(x, 0, t-\tau)\, d\tau$$

$$+ \int_0^t [a^2 \dot{v}(\tau) + z(\tau) F(y(\tau), \tau, \cdots)] G(x, y(\tau), t-\tau)\, d\tau$$

$$+ \int_0^t a^2 z(\tau) v(\tau) \frac{\partial}{\partial \xi} G(x, y(\tau), t-\tau)\, d\tau$$

(1.3.25)

$$+ \int_0^t [a^2 \ddot{\phi}(\xi) + F(\xi, 0, \cdots)] G(x, \xi, t)\, d\xi$$

$$+ \int_0^t d\tau \int_0^{y(\tau)} \frac{d}{d\tau} F(\xi, \tau, \cdots) G(x, \xi, t-\tau)\, d\xi$$

$$- a^2 \int_0^t z(\tau) \dot{\psi}(y(\tau)) \frac{\partial}{\partial \xi} G(x, y(\tau), t-\tau)\, d\tau \equiv U^*.$$

Analogously

$$\begin{aligned}\frac{dQ}{dt} &= \{z(0)\,[\dot\psi(l) - v(0)] - a^2\ddot\phi(l) - F(l,0,\cdots)\}g(x(t),l,t)\\ &+ \int_0^l \left\{\left[a^2\dddot\phi(\xi) + \frac{\partial}{\partial\xi}F(\xi,0,\cdots)\right]g(x(t),\xi,t)\right.\\ &\qquad\left. + \dot x(t)\ddot\phi(\xi)\,G(x(t),\xi,t)\right\}d\xi\\ &+ a^2\int_0^t \frac{d}{d\tau}f(\tau,w(\tau))\frac{\partial}{\partial\xi}g(x(t),0,t-\tau)\,d\tau\\ &- \int_0^t \dot x(t)\frac{d}{d\tau}f(\tau,w(\tau))\,G(x(t),0,t-\tau)\,d\tau\\ &+ \int_0^t \left\{\frac{d}{d\tau}[z(\tau)(\dot\psi(y(\tau)) - v(\tau))]g(x(t),y(\tau),t-\tau)\right.\\ &\qquad\left. + \dot x(t)\dot v(\tau)\,G(x(t),y(\tau),t-\tau)\right\}d\tau\\ &+ \int_0^t \{z(\tau)\,[z(\tau)(\dot\psi(y(\tau)) - v(\tau) - F(y(\tau),\tau,\cdots))] - a^2\dot v(\tau)\}\\ &\qquad\cdot\frac{\partial}{\partial\xi}g(x(t),y(\tau),t-\tau)\,d\tau\\ &+ \int_0^t \dot x(t)\,[z(\tau)(v(\tau) - \dot\psi(y(\tau)) + F(y(\tau),\tau,\cdots))]\\ &\qquad\cdot\frac{\partial}{\partial\xi}G(x(t),\xi,t-\tau)\,d\tau\\ &- \int_0^t d\tau\int_0^{y(\tau)}\left[\frac{\partial}{\partial\tau}F(\xi,\tau,\cdots)\frac{\partial}{\partial\xi}g(x(t),\xi,t-\tau)\right.\\ &\qquad\left. + \dot x(t)\frac{\partial}{\partial\xi}F(\xi,\tau)\frac{\partial}{\partial\xi}G(x(t),\xi,t-\tau)\right]d\xi.\end{aligned} \tag{1.3.26}$$

Here

$$0 < x(t) < y(t) \text{ for } x(t) \not\equiv y(t), \quad \text{or} \quad x(t) \equiv y(t). \tag{1.3.27}$$

We recall that dots above the symbol for a function always denote derivatives of the function with respect to its argument, d/dt is the total derivative operator, and $\partial/\partial t$ is the partial derivative operator. (See the Introduction, (0.2.13).) Using $(1.1.12_2)$ and $(1.1.14_4)$, we obtain

$$dW/dt = W^* \equiv U^*|_{x=0}; \tag{1.3.28}$$

$$dV/dt = V^* = 2Q^*|_{x=y(t)}. \tag{1.3.29}$$

Finally,

$$\begin{aligned}\frac{dQ}{dx} &= \int_0^t \dot{\phi}(\xi)\, G(x,\xi,t)\, d\xi - \int_0^t \frac{d}{d\tau} f(\tau, w(\tau))\, G(x,0,t-\tau)\, d\tau \\ &+ \int_0^t \dot{v}(\tau)\, G(x, y(\tau), t-\tau)\, d\tau \\ &+ \int_0^t \Big\{ z(\tau)\,[v(\tau) - \dot{\psi}(y(\tau))] \\ &\qquad + F(y(\tau),\tau,\cdots)\frac{\partial}{\partial\xi} G(x,y(\tau),t-\tau)\Big\}\, d\tau \\ &- \int_0^t d\tau \int_0^{y(\tau)} \frac{\partial}{\partial\xi} F(\xi,\tau,\cdots)\frac{\partial}{\partial\xi} G(x,\xi,t-\tau)\, d\xi \equiv Q^{**}.\end{aligned} \tag{1.3.30}$$

We now introduce the notation

$$\begin{cases} f^1_{1,i,0} = -\dfrac{d}{d\tau} f(\tau, w(\tau)); \\ f^1_{1,i,1} = \dot{v}(\tau) + a^{-2} z(\tau) F(y(\tau),\tau,\cdots); \\ f^1_{2,i,1} = z(\tau)\,[v(\tau) - \dot{\psi}(y(\tau))]; \\ f^1_{3,i} = a^2 \ddot{\phi}(\xi) + F(\xi,0,\cdots); \quad f^1_{4,i} = \dfrac{d}{d\tau} F(\xi,\tau,\cdots); \end{cases} \tag{1.3.31}$$

$$\begin{cases} f_0 = z(0)\,[\dot{\psi}(l) - v(0)] - a^2\ddot{\phi}(l) - F(l,0,\cdots); \\ f^2_{1,i,0} = (-1)^i a^{-2}\dot{x}(t)\dfrac{d}{d\tau} f(\tau, w(\tau)); \quad f^2_{2,i,0} = \dfrac{d}{d\tau} f(\tau, w(\tau)); \\ f^2_{1,i,1} = a^{-2}\dfrac{d}{d\tau}\,[z(\tau)\dot{\psi}(y(\tau)) - v(\tau))] + (-1)^{i+1} a^{-2}\dot{x}(t)\dot{v}(\tau); \\ f^2_{2,i,1} = a^{-2}[z(\tau) + (-1)^i \dot{x}(t)]\cdot[z(\tau)(\dot{\psi}(y(\tau)) - v(\tau)) \\ \qquad\qquad - F(y(\tau),\tau,\cdots)] - \dot{v}(\tau); \\ f^2_{3,i} = a^2\ddot{\phi}(\xi) + (-1)^{i+1}\dot{x}(t)\ddot{\phi}(\xi) + \dfrac{\partial}{\partial\xi} F(\xi,0,\cdots); \\ f^2_{5,i} = (-1)^i \dot{x}(t)\dfrac{\partial}{\partial\xi} F(\xi,\tau,\cdots) - \dfrac{\partial}{\partial\tau} F(\xi,\tau,\cdots); \end{cases} \tag{1.3.32}$$

$$\begin{cases} f^3_{1,i,0} = -\dfrac{d}{d\tau} f(\tau, w(\tau)); \quad f^3_{1,i,1} = \dot{v}(\tau); \\ f^3_{2,i,1} = z(\tau)\,[v(\tau) - \dot{\psi}(y(\tau))] + F(y(\tau), \tau, \cdots); \\ f^3_{3,i} = \ddot{\varphi}(\xi); \quad f^3_{5,i} = -\dfrac{\partial}{\partial \xi} F(\xi, \tau, \cdots). \end{cases} \tag{1.3.33}$$

Using (0.2.14), (1.3.31)—(1.3.33) as well as (1.1.13), we can reduce (1.3.25), (1.3.26) and (1.3.30) to the form

$$\begin{aligned} \frac{dU}{dt} &\equiv U^*(x,t,\cdots) \\ &= \sum_{i=1}^{2} [I_{1,i}(x,t,0,\tau,a|f^1_{1,i,0}) + I_{1,i}(x,t,y(\tau),\tau,a|f^1_{1,i,1}) \\ &\quad + I_{2,i}(x,t,y(\tau),\tau,a|f^1_{2,i,1}) + I_{3,i}(x,t,\xi,0,l,a|f^1_{3,i}) \\ &\quad + I_{4,i}(x,t,\xi,\tau,0,y(\tau),a|f^1_{4,i})\,]. \end{aligned} \tag{1.3.34}$$

$$\begin{aligned} \frac{dQ}{dt} &\equiv Q^*(x,t,\cdots) \\ &= \sum_{i=1}^{2} (-1)^{i+1} [f_0 E(x(t) + (-1)^i l, a^2 t) + I_{1,i}(x(t),t,0,\tau,a|f^2_{1,i,0}) \\ &\quad + I_{1,i}(x(t),t,y(\tau),\tau,a|f^2_{1,i,1}) + I_{2,i}(x(t),t,0,\tau,a|f^2_{2,i,0}) \\ &\quad + I_{2,i}(x(t),t,y(\tau),\tau,a|f^2_{2,i,1}) + I_{3,i}(x(t),t,\xi,0,l,a|f^2_{3,i}) \\ &\quad + I_{5,i}(x(t),t,\xi,\tau,0,y(\tau),a|f^2_{5,i})\,], \end{aligned} \tag{1.3.35}$$

$$\begin{aligned} \frac{dQ(x,t)}{dx} &\equiv Q^{**}(x,t,\cdots) \\ &= \sum_{i=1}^{2} [I_{1,i}(x,t,0,\tau,a|f^3_{1,i,0}) + I_{1,i}(x,t,y(\tau),\tau,a|f^3_{1,i,1}) \\ &\quad + I_{2,i}(x,t,y(\tau),\tau,a|f^3_{2,i,1}) + I_{3,i}(x,t,\xi,0,l,a|f^3_{3,i}) \\ &\quad + I_{5,i}(x,t,\xi,\tau,0,y(\tau),a|f^3_{5,i})\,]. \end{aligned} \tag{1.3.36}$$

We now estimate the integrals $I_{i,j}$ under the assumption that the argument functions f satisfy the inequality

$$|f| < \frac{L}{\sqrt{t}}. \tag{1.3.37}$$

Obviously

$$|I_{1,i}(x,t,\xi,\tau,a|f)| < \frac{aL}{2\sqrt{\pi}}\int_0^t \frac{d\tau}{\sqrt{\tau}\sqrt{t-\tau}} = \frac{aL\sqrt{\pi}}{2}. \tag{1.3.38}$$

Furthermore, by analogy to (1.3.12) and (1.3.13) we obtain

$$|I_{4,i}(x,t,\xi,\tau,\alpha,\beta,a|f)| < \frac{1}{\sqrt{\pi}}\int_0^t \frac{d\tau}{\sqrt{\tau}}\left|\int_c^d \exp(-\lambda^2)\,d\lambda\right| < 2L\sqrt{t}; \tag{1.3.39}$$

$$|I_{5,i}(x,t,\xi,\tau,\alpha,\beta,a|f)|$$

$$< \frac{L}{a\sqrt{\pi}}\int_0^t \frac{d\tau}{\sqrt{\tau}\sqrt{t-\tau}}\left|\int_c^d \lambda\exp(-\lambda^2)\,d\lambda\right| < \frac{L\sqrt{\pi}}{a}. \tag{1.3.40}$$

We now estimate $I_{2,i}$. Using (1.3.15) we find that

$$|I_{2,i}(x(t),t,\xi(\tau),\tau,a|f)|$$

$$< \frac{LA}{2a\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{\tau}}\frac{|x(t)-\xi(t)|}{(t-\tau)^{3/2}}\exp\left[-\frac{(x(t)-\xi(t))^2}{4a^2(t-\tau)}\right]d\tau \tag{1.3.41$_1$}$$

$$+\frac{L}{4a\sqrt{\pi}}\int_0^t \frac{|\xi(t)-\xi(\tau)|}{\sqrt{\tau}(t-\tau)^{3/2}}\exp\left(-\frac{[\xi(t)-\xi(\tau)]^2}{4a^2(t-\tau)}\right)d\tau = I' + I''.$$

The number A is defined in agreement with (1.3.17). As in the derivation of the estimates (1.3.10)—(1.3.19), we obtain

$$|\xi(t)-\xi(\tau)| < N_5(t-\tau).$$

Hence

$$|I''| < \frac{LN_5}{4a\sqrt{\pi}}\int_0^t \frac{d\tau}{\sqrt{\tau}\sqrt{t-\tau}} = \frac{LN_5\sqrt{\pi}}{4a}. \tag{1.3.41$_2$}$$

Therefore, setting

$$z^2 = \frac{[x(t)-\xi(t)]^2}{4a^2}\left[\frac{1}{t-\tau}-\frac{1}{t}\right];\quad |x(t)-\xi(t)| = \rho \tag{1.3.41$_3$}$$

in I', we find that

$$|I'| < \frac{2AL\epsilon}{\sqrt{\pi}}\cdot\frac{\exp[-\rho^2/4a^2t]}{\sqrt{t}}\cdot\int_0^\infty \exp(-z^2)\,dz$$

$$= \frac{AL\epsilon}{\sqrt{t}}\exp\left(-\frac{\rho^2}{4a^2t}\right). \tag{1.3.41$_4$}$$

Here ϵ is defined in agreement with (1.3.17). Combining $(1.3.41_1)$ and $(1.3.41_4)$, we see that

$$|I_{2,1}(x(t),t,\xi(\tau),\tau,0|f)| < \frac{AL\epsilon}{\sqrt{t}}\exp\left(-\frac{\rho^2}{4a^2t}\right)+\frac{LN_5\sqrt{\pi}}{4a}. \tag{1.3.41}$$

In exactly the same way, we obtain

$$|I_{2,2}(x(t),t,\xi,\tau,a|f)| < L\left[\frac{A\exp(-\rho^{*2}/4a^2t)}{\sqrt{t}}+N_5\frac{\sqrt{\pi}}{4a}\right]. \tag{1.3.42}$$

Here

$$\rho^* = |x(t)+\xi(t)|. \tag{1.3.43}$$

Comparing the estimates (1.3.38)—(1.3.42), the definitions (1.3.31)—(1.3.33) and the inequalities (1.1.9), (1.1.10) and (1.3.1), and reasoning as in the proof of (1.3.23), we can prove that there exist numbers $L_i > 0$ $(i = 1,\cdots,5)$ and $L^* > 0$ so large, and $T \leqq T_0$ so small, that the following inequalities hold:

$$\begin{gathered}
|w| < N_1;\quad |u| < N_2;\quad |v| < N_3;\quad |q| < N_4;\quad |z| < N_5;\\
|\dot{w}| < \frac{L_1}{\sqrt{t}};\quad \left|\frac{\partial u}{\partial t}\right| < \frac{L_2}{\sqrt{t}};\quad |v| < \frac{L_3}{\sqrt{t}};\quad \left|\frac{\partial q}{\partial t}\right| < \frac{L_4}{\sqrt{t}};\\
|\dot{z}| < \frac{L_5}{\sqrt{t}};\quad \left|\frac{\partial q}{\partial x}\right| < L^* \text{ for } 0 \leqq t \leqq T;\quad 0 \leqq x;\quad \xi < 1;
\end{gathered} \tag{1.3.44}$$

moreover, from these estimates we infer similar estimates for W, U, Q, V and Z. But together with (1.3.23) this proves the existence of a $T > 0$ so small that for $0 \leqq t \leqq T$ the sequences $\{w_n(t)\},\cdots,\{z_n(t)\}$ are uniformly bounded and equicontinuous; hence the proof is complete, since there are then subsequences $\{w_n(t)\},\cdots,\{z_n(t)\}$ which converge and hence prove the existence of a solution to the system of integral equations of the problem.

§4. Uniqueness of the solution

A. Up to now we have proved the existence in the small of a solution to the system of integral equations of the problem. However, it is possible to prove yet stronger assertions. Thus, for example, we will prove that the sequences $\{w_n\}$, $\{u_n\}$, $\{q_n\}$, $\{v_n\}$ and $\{z_n\}$ are not only uniformly bounded and equicontinuous but are also uniformly convergent. This will imply convergence of the entire iteration sequence to a solution of the system of integral equations of the problem.

It suffices to prove uniform convergence of the series

$$\sum_{m=0}^{\infty}(u_{m+1}-u_m),\cdots,\sum_{m=0}^{\infty}(z_{m+1}-z_m). \tag{1.4.1}$$

This is guaranteed for $0 \leqq t \leqq T_1 < 1/L$ if

$$|u_{m+1}-u_m| < M(LT_1)^{m/2},\cdots,|z_{m+1}-z_m| < M(LT_1)^{m/2}; \quad m=0,1,2,\cdots, \tag{1.4.2}$$

where M and L are positive constants independent of m.

By virtue of the uniform boundedness of $\{u_n\},\cdots,\{z_n\}$, (1.4.2) holds for $m=0$. Hence it suffices to show that if (1.4.2) is valid for $m=n$, then it is equally valid for $m=n+1$. With this aim we examine the family of functions

$$\{f(x,t,\xi,\tau)\};\quad \{x(t)\};\quad \{\xi(\tau)\};\quad \{y(t)\},$$

having the following properties:

a) $x(t)$, $\xi(t)$ and $y(t)$ are defined and continuously differentiable functions for $0 \leqq t \leqq T$, and have continuous second derivatives for $0 \leqq t \leqq T$; moreover

$$\begin{gathered} 0 \leqq y(t) \leqq 1;\quad 0 \leqq x(t) \leqq 1;\quad 0 \leqq \xi(t) \leqq 1 \text{ for } 0 \leqq t \leqq T; \\ |z(t)| \leqq N \quad (z \equiv \dot{y});\quad |\dot{\xi}(t)|,\quad |\dot{x}(t)| \leqq N; \\ |\sqrt{t}\dot{z}(t)|;\quad |\ddot{\xi}(t)\sqrt{t}|;\quad |\ddot{x}(t)\sqrt{t}| \leqq L. \end{gathered} \tag{1.4.3}$$

b) $f(x,t,\xi,\tau)$ is defined for $0 \leqq \tau \leqq t \leqq T$, and continuous together with its partial derivatives with respect to all of its arguments for $0 < \tau \leqq t \leqq T$; moreover

$$|f| < M;\quad \left|\sqrt{t}\frac{\partial f}{\partial x}\right|;\quad \left|\sqrt{t}\frac{\partial f}{\partial t}\right|;\quad \left|\sqrt{\tau}\frac{\partial f}{\partial \tau}\right|;\quad \left|\sqrt{\tau}\frac{\partial f}{\partial \xi}\right| < M^*, \tag{1.4.4}$$

where M and M^* are values of the same order of magnitude, and the inequalities are uniform for the entire family under investigation.

We will investigate variations of the integral within the class of functions being considered. Let f, $x(t)$, $\xi(\tau)$, $y(t)$ and f^*, $x^*(t)$, $\xi^*(\tau)$, $y^*(t)$ be two groups of admissible functions. We will write

$$\begin{aligned} &I_{i,j}(x^*(t),t,\xi^*(\tau),\tau,\cdots|f^*(x^*(t),t,\xi^*(\tau),\tau)) \\ &\quad - I_{i,j}(x(t),t,\xi(\tau),\tau,\cdots|f(x(t),t,\xi(\tau),\tau)) \\ &\quad = \delta I_{i,j}(x(t),t,\xi(\tau),\tau,\cdots|f(x(t),t,\xi(\tau),\tau)) \end{aligned} \tag{1.4.5}$$

and

$$\max|\delta I_{i,j}| = \delta^* I_{i,j}; \quad 0 \leqq \tau \leqq t \leqq T; \quad 0 \leqq x, \xi \leqq 1; \quad i = 1, 2, \cdots, 5; \quad j = 1, 2. \tag{1.4.6}$$

We will use analogous notation for all functions in the class of functions under investigation.

For any i and j we have

$$\delta I_{i,j} = I_{i,j}(x^*(t), t, \xi^*(\tau), \tau, \cdots | \delta f) + \Delta I_{i,j}, \tag{1.4.7}$$

where

$$\begin{aligned}\Delta I_{i,j} = {} & I_{i,j}(x^*(t), t, \xi^*(\tau), \tau, \cdots | f(x(t), t, \cdots)) \\ & - I_{i,j}(x(t), t, \xi(\tau), \tau, \cdots | f(x(t), t, \cdots)).\end{aligned} \tag{1.4.8}$$

We find that

$$\Delta I_{1,i} = a^2 \int_0^t f d\tau \int_{x(t)+(-1)^i \xi(\tau)}^{x^*(t)+(-1)^i \xi^*(\tau)} \frac{\partial}{\partial \zeta} E(\zeta, a^2(t-\tau)) d\zeta,$$

or, equivalently,

$$\begin{aligned}\Delta I_{1,i} = {} & a^2 \int_0^t f d\tau \int_0^{\delta[x(t)+(-1)^i \xi(t)]} \frac{\partial}{\partial \zeta} E(x(t) + (-1)^i \xi(\tau) + \zeta, a^2(t-\tau)) d\zeta \\ & + a^2 \int_0^t f d\tau \int_0^{(-1)^{i+1} \delta[\xi(t)-\xi(\tau)]} \frac{\partial}{\partial \zeta} E(x^*(t) + (-1)^i \xi(\tau) \\ & \qquad + (-1)^i \delta\xi(t) + \zeta, a^2(t-\tau)) d\zeta \\ = {} & \Delta I'_{1,i} + \Delta I''_{1,i}.\end{aligned} \tag{1.4.9}$$

We now set

$$\alpha_m = \max_{0 \leqq \lambda < \infty} \lambda^m \exp(-\lambda^2). \tag{1.4.10}$$

By (1.4.3) we have

$$\left|\frac{\partial E}{\partial \zeta}\right| < \frac{\alpha_1}{2a^2\sqrt{\pi}} \frac{1}{(t-\tau)}; \quad |\delta[\xi(t) - \dot{\xi}(\tau)]| < \delta^* \xi(\tau)(t-\tau). \tag{1.4.11}$$

Consequently

$$|\Delta I''_{1,i}| < \frac{M\alpha_1}{2\sqrt{\pi}} t \delta^* \dot{\xi}(\tau). \tag{1.4.12}$$

Furthermore, changing the order of integration for $x(t) \neq \xi(t)$ in the term $\Delta I'_{1,i}$ (in the case $x(t) \not\equiv \xi(t)$), we find that

$$\Delta I'_{1,i} = (-1)^i a^2 \int_0^{\delta[x(t)+(-1)^i\xi(t)]} d\zeta \times \int_0^t f \frac{\partial}{\partial \xi} E(x(t) + (-1)^i \xi(t) + \zeta, a^2(t-\tau)) d\tau,$$

or, in the notation of (0.2.14),

$$\Delta I'_{1,i} = (-1)^i \int_0^{\delta[x(t)+(-1)^i\xi(t)]} I_{2,i}(x(t) + \zeta, t, \xi(t), \tau, o | f(x(t), t, \xi(\tau), \tau)) \, d\zeta. \tag{1.4.13}$$

Using estimates (1.3.18) and (1.3.19) together with the theorem of the mean, and observing that the constants in these estimates do not depend on δx and $\delta \xi$, we find that

$$|\Delta I'_{1,i}| < M'[|\delta x(t)| + \delta \xi(t)|], \tag{1.4.14}$$

where M' depends only on M, A, N and T. Combining (1.4.12) and (1.4.14), we obtain

$$|\Delta I_{1,i}| < M_{1,1} t \delta^* \xi + M_{1,2}(\delta^* x + \delta^* \xi). \tag{1.4.15}$$

We now examine $\Delta I_{2,i}$. Setting

$$\begin{aligned} z^*(t,\tau) &= x^*(t) + (-1)^i \xi^*(\tau); \\ z(t,\tau) &= x(t) + (-1)^i \xi(\tau), \end{aligned} \tag{1.4.16}$$

we see that

$$\begin{aligned} &\frac{z(t,\tau)}{4a\sqrt{\pi}(t-\tau)^{3/2}} \exp\left[-\frac{z^2(t,\tau)}{4a^2(t-\tau)}\right] \\ &= -\frac{1}{2}\left[\frac{d}{d\tau} \operatorname{erfc} \frac{z(t,\tau)}{2a\sqrt{t-\tau}} + 2E(z(t,\tau), a^2(t-\tau)) \frac{\partial z(t,\tau)}{\partial \tau}\right]. \end{aligned} \tag{1.4.17}$$

Consequently

$$\begin{aligned} \Delta I_{2,i} = &-\frac{1}{2}\int_0^t f \frac{d}{d\tau} \delta \operatorname{erfc} \frac{z(t,\tau)}{2a\sqrt{t-\tau}} d\tau \\ &- \int_0^t f \delta \left(\frac{\partial z(t,\tau)}{\partial \tau}\right) E(z^*(t,\tau), a^2(t-\tau)) \, d\tau \\ &- \int_0^t f \frac{\partial z(t,\tau)}{\partial \tau} \delta E(z(t,\tau), a^2(t-\tau)) \, d\tau \\ = &\Delta I'_{2,i} + \Delta I''_{2,i} + \Delta I'''_{2,i}. \end{aligned} \tag{1.4.18}$$

We have

$$\Delta I''_{2,i} = I_{1,i}\left[x^*(t), t, \xi^*(\tau), \tau, a \,|\, f\cdot\delta\left(\frac{\partial z(t,\tau)}{\partial\tau}\right)\right];$$

(1.4.19)

$$\Delta I'''_{2,i} = \Delta I_{1,i}\left(x(t), t, \xi(\tau), \tau, a \,|\, f\frac{\partial z(t,\tau)}{\partial\tau}\right).$$

Using (1.3.11), (1.4.15), (1.4.16) and (1.4.4), we find that

$$|\Delta I''_{2,i}| < \frac{aM}{\sqrt{\pi}}\sqrt{t}\delta^*\xi;$$

(1.4.20)

$$|\Delta I'''_{2,i}| < M'_1 t\delta^*\dot{\xi} + M'_2(\delta^* x + \delta^*\xi),$$

where M'_1 and M'_2 are suitably chosen constants, independent of $\delta^*\dot{\xi}$, $\delta^* x$ and $\delta^*\xi$.

Furthermore, for $z(t,0) \not\equiv 0$

$$\Delta I'_{2,i} = \frac{1}{2} f(x(t), t, \xi(0), 0)\delta\,\mathrm{erfc}\frac{z(t,0)}{2a\sqrt{t}}$$

(1.4.21)

$$+\frac{1}{2}\int_0^t \frac{d}{d\tau} f(x(t), t, \xi(\tau), \tau)\delta\,\mathrm{erfc}\frac{z(t,\tau)}{2a\sqrt{t-\tau}}\,d\tau.$$

But

$$\frac{1}{2}\delta\,\mathrm{erfc}\frac{z(t,\tau)}{2a\sqrt{t-\tau}} = \int_0^{-\delta z(t,\tau)} E(\zeta + z(t,\tau), a^2(t-\tau))\,d\zeta. \tag{1.4.22}$$

Consequently

$$\frac{1}{2}\left|\delta\,\mathrm{erfc}\frac{z(t,\tau)}{2a\sqrt{t-\tau}}\right| < \frac{1}{2a\sqrt{\pi}}\frac{|\delta z(t,\tau)|}{\sqrt{t-\tau}}. \tag{1.4.23}$$

Comparing (1.4.21), (1.4.22), (1.4.23), (1.4.16) and (1.4.4), we find that

$$|\Delta I'_{2,i}| < \frac{M}{2a\sqrt{\pi t}}[\delta^* x + |\delta\xi(0)|] + \frac{M^*\sqrt{\pi}}{2a}(\delta^* x + \delta^*\xi). \tag{1.4.24}$$

Combining (1.4.24), (1.4.20) and (1.4.18), we obtain

$$|\Delta I_{2,i}| < M_{2,1}\sqrt{t}\delta^*\dot{\xi} + M_{2,2}t^{-1/2}[\delta^* x + |\delta\xi(0)|], \tag{1.4.25}$$

where the $M_{2,i}$ $(i = 1, 2)$ are independent of $\delta^*\xi$, $\delta^* x$ and $\delta^*\dot{\xi}$.

We now examine $\Delta I_{3,i}$. We have

$$\Delta I_{3,i} = \int_{\alpha}^{\beta} f(x(t), t, \xi)\, d\xi \int_{0}^{\delta x(t)} (-1)^i \frac{\partial}{\partial \xi} E(x(t) + \zeta + (-1)^i \xi, a^2 t)\, d\zeta. \tag{1.4.26}$$

On changing the order of integration and taking absolute values, we obtain

$$|\Delta I_{3,i}| < \frac{\alpha_1 M \delta^* x}{\sqrt{\pi t}}, \tag{1.4.27}$$

where α_1 is defined by (1.4.10).

We now examine $\Delta I_{4,i}$. We have

$$\begin{aligned} &\Delta I_{4,i}(x(t), t, \xi, \tau, 0, y(\tau), a | f(x(t), t, \xi, \tau)) \\ &\quad = \int_0^t d\tau \int_0^{y(\tau)} f(x(t), t, \xi, \tau)\, d\xi \\ &\qquad \cdot \int_0^{\delta x(t)} (-1)^i \frac{\partial}{\partial \xi} E(x(t) + (-1)^i \xi + \zeta, a^2(t-\tau))\, d\zeta \\ &\quad + \int_0^t d\tau \int_{y(\tau)}^{y^*(\tau)} f(x(t), t, \xi, \tau) E(x^*(t) + (-1)^i \xi, a^2(t-\tau))\, d\xi \\ &\quad = \Delta I'_{4,i} + \Delta I''_{4,i}. \end{aligned} \tag{1.4.28}$$

By changing the order of integration and using (1.3.13), we find that

$$|\Delta I'_{4,i}| < \frac{M}{a\sqrt{\pi}} \sqrt{t} \delta^* x. \tag{1.4.29}$$

Furthermore, it is clear that

$$|\Delta I''_{4,i}| < \frac{M}{a\sqrt{\pi}} \sqrt{t} \delta^* y. \tag{1.4.30}$$

Combining (1.4.29) and (1.4.30) yields

$$|\Delta I_{4,i}| < \frac{M}{a\sqrt{\pi}} \sqrt{t} (\delta^* x + \delta^* y). \tag{1.4.31}$$

We finally turn to the estimation of $\Delta I_{5,i}$. We have

$$\Delta I_{5,i}(x(t), t, \xi, \tau, 0, y(\tau), a | f(x(t), t, \xi, \tau)) = \Delta I'_{5,i} + \Delta I''_{5,i}, \tag{1.4.32}$$

where

$$\Delta I'_{5,i} = \int_0^t d\tau \int_0^{y(\tau)} f(x(t), t, \xi, \tau)\delta \frac{\partial}{\partial \xi} E(x(t) + (-1)^i \xi, a^2(t-\tau))\, d\xi;$$

(1.4.32$_1$)

$$\Delta I''_{5,i} = \int_0^t d\tau \int_{y(\tau)}^{y^*(\tau)} f(x(t), t, \xi, \tau) \frac{\partial}{\partial \xi} E(x(t) + (-1)^i \xi, a^2(t-\tau))\, d\xi.$$

(1.4.32$_2$)

It is obvious that $\Delta I''_{5,i}$ is estimated as was $\Delta I'_{1,i}$. Thus

$$|\Delta''_{5,i}| < M''_{5,i}\delta^* y, \tag{1.4.33}$$

where $M''_{5,i}$ is a suitably chosen constant.

We now turn to $\Delta I'_{5,i}$. Integrating by parts, we find that

$$\begin{aligned}\Delta I'_{5,i} = &\int_0^t f(x(t), t, y(\tau), \tau)\delta E(x(t) + (-1)^i y(\tau), a^2(t-\tau))\, d\tau \\ &- \int_0^t f(x(t), t, 0, \tau)\delta E(x(t) + (-1)^i l, a^2(t-\tau))\, d\tau \\ &+ \int_0^t d\tau \int_0^{y(\tau)} \frac{\partial}{\partial \xi} f(x(t), t, \xi, \tau)\delta E(x(t) + (-1)^i \xi, a^2(t-\tau))\, d\xi \\ &= \Delta I^{1,1}_{5,i} + \Delta I^{1,2}_{5,i} + \Delta I^{1,3}_{5,i}.\end{aligned} \tag{1.4.34}$$

Here $l = y(0)$.

We note that

$$\Delta I^{1,1}_{5,i} = \Delta I_{1,i}(x(t), t, y(\tau), \tau, a | a^{-2} f(x(t), t, y(\tau), \tau));$$
$$\Delta I^{1,2}_{5,i} = \Delta I_{1,i}(x(t), t, l, \tau, a | a^{-2} f(x(t), t, l, \tau)),$$

and by virtue of the estimate (1.4.15) we obtain

$$|\Delta I^{1,1}_{5,i}| < M^{1,1}_{5,i}[\delta^* x + \delta^* y]; \quad |\Delta I^{1,2}_{5,i}| < M^{1,2}_{5,i}\delta^* x, \tag{1.4.35}$$

where $M^{1,j}_{5,i}$ are particular constants independent of the variation.

We finally examine $\Delta I^{1,3}_{5,i}$. As earlier, we have

$$|\Delta I^{1,3}_{5,i}| \leqq \int_0^t d\tau \int_0^{y(\tau)} \left|\frac{\partial f}{\partial \xi}\right| d\xi \int_0^{|\delta x(t)|} \frac{\partial}{\partial \zeta} E(x(t) + (-1)^i \xi + \zeta, a^2(t-\tau))\, d\zeta.$$

(1.4.36)

Changing the order of integration and using (1.4.4) and (1.3.40), we see that

$$|\Delta I_{5,i}^{1,3}| < \frac{\sqrt{\pi}M^*}{a}\delta^* x. \tag{1.4.37}$$

Combining the above estimates, we obtain

$$|\Delta I_{5,i}| < M_{5,i}[\delta^* x + \delta^* y]. \tag{1.4.38}$$

Using the estimates (1.4.15), (1.4.25), (1.4.26), (1.4.31), (1.4.38) and observing that the first term of (1.4.7) is estimated by means of (1.3.10)—(1.3.13), (1.3.18), (1.3.19), we see that if we replace M by $\delta^* f$, then[6]

$$\begin{aligned}
&\delta^* I_{1,i} < K_1'\sqrt{t}\delta^* f_{1,i} + M_{1,1}t\delta^*\dot{\xi} + M_{1,2}(\delta^* x + \delta^*\xi);\\
&\delta^* I_{2,i} < K_2'\left[\epsilon\,\mathrm{erfc}\frac{|x(t) + (-1)^i\xi(t)|}{2a\sqrt{t}} + \sqrt{t}\right]\delta^* f_{2,i}\\
&\qquad + M_{2,1}\sqrt{t}\delta^*\dot{\xi} + \frac{M_{2,2}}{\sqrt{t}}[\delta^* x + \delta^*\xi(0)];\\
&\delta^* I_{3,i} < \delta^* f_{3,i} + M_{3,i}\frac{\delta^* x}{\sqrt{t}};\\
&\delta^* I_{4,i} < K_4' d\delta^* f_{4,i} + M_4\sqrt{t}(\delta^* x + \delta^* y);\\
&\delta^* I_{5,i} < K_5'\sqrt{t}\delta^* f_{5,i} + M_5(\delta^* x + \delta^* y).
\end{aligned} \tag{1.4.39}$$

Here K_i, $M_{i,j}$ and M_i are constants independent of $\delta x, \cdots, \delta f_{5,i}$.

We will assume that

$$\begin{aligned}
&x(t) \equiv y(t), \text{ or } x(t) \equiv x = \text{const} \in (0, y(t));\\
&\xi(t) \equiv y(t), \text{ or } \xi(t) \equiv 0;\\
&\delta y(0) = 0.
\end{aligned} \tag{1.4.40}$$

Then for $0 \leqq t \leqq T$ and $|z(t)| < N$

$$\delta^* x \leqq t\delta^* z; \quad \delta^*\xi \leqq t\delta^* z; \quad \delta^* y \leqq t\delta^* z. \tag{1.4.41}$$

In what follows the symbols K, $\widetilde{K}$, etc. denote arbitrary constants entering into the estimates, which are independent of the variation. Then (1.4.39) can be written in the form

$$\begin{aligned}
&\delta^* I_{i,j} < K\sqrt{t}(\delta^* f_{i,j} + \delta^* z) \quad i = 1,4,5; \quad j = 1,2;\\
&\delta^* I_{2,i} < K\left\{\left[\epsilon\,\mathrm{erfc}\frac{|x(t) + (-1)^i\xi(t)|}{2a\sqrt{t}} + \sqrt{t}\right]\delta^* f_{2,i} + \sqrt{t}\delta^* z\right\};\\
&\delta^* I_{3,i} < \delta^* f_{3,i} + K\sqrt{t}\delta^* z.
\end{aligned} \tag{1.4.42}$$

[6] Here we assign the same indices to f as we assigned to the operator $I_{i,j}$.

Here ϵ is defined by means of (1.3.17).

We now estimate variations of the functionals U, W, Q, V and Z which are induced by variations of w, u, q, v, y and z. By virtue of (1.3.4), (1.3.5), (1.1.10) and (1.4.42) we have[7]

$$
\begin{aligned}
&\delta^*U < K\sqrt{t}[\delta^*w + \delta^*u + \delta^*q + \delta^*v + \delta^*z];\\
&\delta^*W < K\sqrt{t}[\delta^*w + \delta^*u + \delta^*q + \delta^*v + \delta^*z];\\
&\delta^*Q < K\sqrt{t}[\delta^*u + \delta^*q + \delta^*z] + \widetilde{K}(\delta^*w + \delta^*v);\\
&\delta^*V < K\sqrt{t}[\delta^*w + \delta^*u + \delta^*q + \delta^*v + \delta^*z];\\
&\delta^*Z < K\sqrt{t}\delta^*z + \widetilde{K}\delta^*v.
\end{aligned}
\tag{1.4.43}
$$

Let $0 \leqq \tau \leqq t$, $0 \leqq \xi \leqq 1$. We obtain

$$
\begin{aligned}
&\delta^*w = \delta_n w = \max|w_n(\tau) - w_{n-1}(\tau)|;\\
&\delta^*u = \delta_n u = \max|u_n(\xi,\tau) - u_{n-1}(\xi,\tau)|;\\
&\delta^*v = \delta_n v = \max|v_n(\tau) - v_{n-1}(\tau)|;\\
&\delta^*q = \delta_n q = \max|q_n(\xi,\tau) - q_{n-1}(\xi,\tau)|;\\
&\delta^*z = \delta_n z = \max|z_n(\tau) - z_{n-1}(\tau)|.
\end{aligned}
\tag{1.4.44}
$$

The definitions (1.1.15) and estimates (1.4.43) imply

$$
\begin{aligned}
&\delta_{n+1}w < K\sqrt{t}(\delta_n w + \delta_n u + \delta_n q + \delta_n v + \delta_n z);\\
&\delta_{n+1}u < K\sqrt{t}(\delta_{n+1}w + \delta_n u + \delta_n q + \delta_n v + \delta_n z);\\
&\delta_{n+1}v < K\sqrt{t}(\delta_{n+1}w + \delta_{n+1}u + \delta_n q + \delta_n v + \delta_n z);\\
&\delta_{n+1}q < K\sqrt{t}(\delta_{n+1}u + \delta_n q + \delta_n z) + \widetilde{K}(\delta_{n+1}w + \delta_{n+1}v);\\
&\delta_{n+1}z < K\sqrt{t}\delta_n z + \delta_{n+1}v.
\end{aligned}
\tag{1.4.45}
$$

Hence there obviously exists some $T' > 0$ $(T' \leqq T)$ sufficiently small and L sufficiently large, so that if

$$
\delta_n = \max\{\delta_n w;\ \delta_n u;\ \delta_n v;\ \delta_n q;\ \delta_n z\},
\tag{1.4.46}
$$

then for $0 \leqq t \leqq T'$

$$
\delta_{n+1} < \sqrt{Lt}\cdot\delta_n.
\tag{1.4.47}
$$

But the inequalities (1.4.47) and (1.4.2) are clearly equivalent, which completes the proof.

---

[7] We note that the expressions $I_{3,j}$ contain arguments ϕ or $\dot{\phi}$ which are not being varied. In the calculation of the variation of Z we write $Z(t, \psi(y), v, y) = Z(t, v, y)$.

B. Let

$$u = \lim_{n\to\infty} u_n; \quad w = \lim_{n\to\infty} w_n; \quad q = \lim_{n\to\infty} q_n;$$
$$v = \lim_{n\to\infty} v_n; \quad z = \lim_{n\to\infty} z_n; \quad y = \lim_{n\to\infty} y_n. \tag{1.4.48}$$

By the uniform boundedness of the sequences

$$\left\{\sqrt{t}\frac{\partial u_n}{\partial t}\right\}; \quad \left\{\sqrt{t}\frac{dw_n}{dt}\right\}; \quad \left\{\sqrt{t}\frac{\partial q_n}{\partial t}\right\}; \quad \left\{\sqrt{t}\frac{dv_n}{dt}\right\}; \quad \left\{\sqrt{t}\frac{dz_n}{dt}\right\}; \quad \left\{\frac{\partial q_n}{\partial x}\right\}$$

it follows that $u, \cdots, z$ satisfy a uniform Lipschitz condition of the form (1.1.17) for $0 \leqq t \leqq T'$. This shows that $F(t,x,u,q,y,z)$ has the degree of smoothness needed for the application of the equivalence theorem proved in §2. Thus it is possible to assert that the solution constructed for the system of integral equations is simultaneously the solution to the original boundary problem (1.1.2)—(1.1.6) satisfying in addition the conditions (1.1.17).

It now remains for us to deal with the problem of uniqueness. The uniqueness of a solution (u,w,q,v,z,y) is clear in the class of functions satisfying conditions (1.3.1) and (1.3.2), i.e. functions differentiable with respect to x and t for $t > 0$, whose derivatives with respect to t grow not faster than $t^{-1/2}$ as $t \to 0$. In fact, for such functions the estimate (1.4.43) is valid. Let $(u, \cdots, y)$ and $(u^*, \cdots, y^*)$ be two admissible systems of solutions. Then, since

$$\delta^*U = \delta^*u; \quad \delta^*W = \delta^*w; \quad \delta^*Q = \delta^*q; \quad \delta^*V = \delta^*v; \quad \delta^*Z = \delta^*z, \tag{1.4.49}$$

(1.4.43) can be written in the form

$$\delta^*U, \quad \delta^*W, \quad \delta^*Q, \quad \delta^*V, \quad \delta^*Z < K\sqrt{t}\delta, \tag{1.4.50}$$

where $K > 0$ is some constant and

$$\delta = \max\{\delta^*w; \delta^*u; \delta^*q; \delta^*v; \delta^*z\};$$
$$0 \leqq x \leqq \min(y, y^*); \quad 0 \leqq \tau \leqq t. \tag{1.4.51}$$

Comparing (1.4.49) and (1.4.51) we see that for $t > 0$ sufficiently small and $\delta > 0$ we have $\delta < K\sqrt{t}\delta < \delta$, which is impossible.

Obviously by using the method of continuation of the solution it is possible to extend this result to any interval $(0, T)$ in which the existence of a solution has been proved.

Nevertheless, one cannot be satisfied with this proof of uniqueness; for the existence of a solution was established only in a wider class of

functions than that in which the uniqueness theorem was derived. Therefore the problem of uniqueness of a solution requires a special investigation.

We note first of all that for the proof of uniqueness it suffices to establish the validity of the following inequalities. As above, let $(u, \cdots, z)$ and $(u^*, \cdots, z^*)$ be two systems of solutions to the problem. Furthermore, let $(U, \cdots, Z)$ and $(U^*, \cdots, Z^*)$ be the values of the functionals $U, \cdots, Z$ corresponding to them. We now set

$$\bar{y}(t) = \min(y(t), y^*(t)) \tag{1.4.52}$$

and assume that the following inequalities are actually true:

$$\begin{aligned} &|U(x,t,\cdots) - U^*(x,t,\cdots)| < N(\rho)L(t)\delta; \\ &|Q(x,t,\cdots) - Q^*(x,t,\cdots)| < N(\rho)L(t)\delta; \\ &|W(t,\cdots) - W^*(t,\cdots)| < ML(t)\delta; \\ &|V(t,\cdots) - V^*(t,\cdots)| < ML(t)\delta; \\ &|Z(t,\cdots) - Z^*(t,\cdots)| < ML(t)\delta, \end{aligned} \tag{1.4.53}$$

where N and L are defined for $0 \leqq t \leqq T$, such that

$$\lim_{\rho \to 0} N(\rho) = \infty; \quad \lim_{t \to +0} L(t) = 0, \tag{1.4.53*}$$

M is a constant independent of δ, and

$$\delta = \max\{|w - w^*|; |v - v^*|; |z - z^*|; |u - u^*|; |q - q^*|\} \tag{1.4.54}$$

for

$$0 \leqq t \leqq T; \quad 0 < \rho \leqq x \leqq \bar{y}(t) - \rho < \bar{y}(t).$$

Fixing $\rho > 0$ as an arbitrarily small value, we again find that for $0 \leqq t \leqq T(\rho)$ and $\rho \leqq x \leqq \bar{y} - \rho$ we have

$$u = u^*; \quad q = q^*; \quad w = w^*; \quad v = v^*; \quad z = z^*. \tag{1.4.55}$$

Considering $T(\rho)$ as a new initial moment in time, we prove the validity of (1.4.55) in the interval $(0, 2T(\rho))$. If we continue with this reasoning we see that (1.4.55) is valid on every interval $(0, T)$ of definition of the solutions $(u, \cdots, z)$ and $(u^*, \cdots, z^*)$ for $\rho \leqq x \leqq \bar{y}(t) - \rho$ $(\bar{y} = \min(y, y^*))$. Since $\rho > 0$ is arbitrarily small and u and q are continuous for $0 \leqq x \leqq y(t)$, we see that the conditions of the uniqueness theorem are satisfied in every interval $(0, T)$ in which the solution is defined.

We now prove that the estimate (1.4.53) is actually valid in the class of functions satisfying a Lipschitz condition of the form

$$|f(t)-f(\tau)|<A\frac{|t-\tau|}{\sqrt{t}+\sqrt{\tau}},\quad A>0, \tag{1.4.56}$$

with respect to t. To this end we note that differentiability of the functions u, w, q, v and z with respect to t is needed only to be able to integrate by parts in order to estimate the integrals $I_{2,i}$ and $I_{5,i}$. Therefore in this case the estimates (1.4.15), (1.4.27) and (1.4.31) remain valid. Thus it remains for us only to investigate the terms $\Delta I_{2,i}$ and $\Delta I_{5,i}$.[8)]

We note first that

$$\Delta I_{2,i}(x,t,0,\tau|f)\equiv 0 \text{ for } x\equiv \text{const}. \tag{1.4.57}$$

Furthermore, it is obvious that for $0\leqq x\leqq y(t)-\rho<y(t)$ we have

$$|\Delta I_{2,i}(x,t,y(\tau),\tau,a|f)|<N(\rho)\delta^* y \quad (i=1,2), \tag{1.4.58}$$

since within this interval the expression $xE(x+(-1)^i y(\tau),\alpha^2(t-\tau))$ together with its partial derivatives with respect to x and t is bounded.

By the same reasoning we find that for $y(\tau)\geqq\rho_0>0$ and $0\leqq\tau\leqq t\leqq T$ we have

$$\Delta I_{2,2}(y(t),t,y(\tau),\tau,a|f)<M\delta^* y. \tag{1.4.59}$$

We now examine $\Delta I_{2,i}(y(t),t,y(\tau),\tau,a|f)$. We have

$$\begin{aligned}&\Delta I_{2,1}(y(t),t,y(\tau),\tau,a|f(\tau))\\&\quad=f(t)\Delta I_{2,1}(y(t),t,y(\tau),\tau,a|1)\\&\qquad+\Delta I_{2,1}(y(t),t,y(\tau),\tau,a|f(\tau)-f(t))\\&\quad=\Delta I'_{2,1}+\Delta I''_{2,1}.\end{aligned} \tag{1.4.60}$$

The term $\Delta I'_{2,1}$ is estimated by using (1.4.25), setting $\xi(\tau)\equiv y(\tau)$ and $x(t)\equiv y(t)$. It remains for us to estimate $\Delta I''_{2,i}$. We have

$$\begin{aligned}\Delta I''_{2,1}=&\int_0^t\frac{f(\tau)-f(t)}{4a\sqrt{\pi}\sqrt{t-\tau}}\\&\cdot\frac{\mu^*(t,\tau)\exp\left(-\dfrac{\mu^{*2}(t,\tau)}{4a^2(t-\tau)}\right)-\mu(t,\tau)\exp\left(-\dfrac{\mu^2(t,\tau)}{4a^2(t-\tau)}\right)}{t-\tau}d\tau,\end{aligned} \tag{1.4.61}$$

[8)] In evaluating the variations of the functions $f_{i,j}$ emanating from the expressions for the operators $I_{i,j}$ we are not required to differentiate. Thus it is sufficient to investigate $\Delta I_{i,j}$ instead of $\delta I_{i,j}$.

where

$$\mu(t,\tau) = y(t) - y(\tau) = \int_\tau^t z(\lambda)\,d\lambda;$$

(1.4.62)

$$\mu^*(t,\tau) = y^*(t) - y^*(\tau) = \int_\tau^t z^*(\lambda)\,d\lambda.$$

Let[9)]

(1.4.63) $$|f(\tau) - f(t)| < A\sqrt{t-\tau}; \quad |z|, \quad |z^*| < N.$$

We find that

$$\begin{aligned}\Delta I''_{2,1} &= I' + I'' \\ &= \int_0^t \frac{f(\tau)-f(t)}{4a\sqrt{\pi}\sqrt{t-\tau}} \frac{\mu^*(t,\tau)-\mu(t,\tau)}{t-\tau} \exp\left(-\frac{\mu^{*2}(t,\tau)}{4a^2(t-\tau)}\right) d\tau \\ &\quad + \int_0^t \frac{f(\tau)-f(t)}{4a\sqrt{\pi}\sqrt{t-\tau}} \frac{\mu(t,\tau)\,d\tau}{t-\tau} \\ &\qquad \cdot \int_{\mu(t,\tau)}^{\mu^*(t,\tau)} \frac{\zeta}{2a^2(t-\tau)} \exp\left(-\frac{\zeta^2}{4a^2(t-\tau)}\right) d\zeta.\end{aligned} \tag{1.4.64}$$

Estimating the absolute values with the help of (1.4.63), we find that

$$|I'| < \frac{At\delta^* z}{4a\sqrt{\pi}}; \quad |I''| < \frac{A}{4a^2\sqrt{\pi}} t^{3/2}\delta^* z,$$

while at the same time

(1.4.65) $$|\Delta I''_{2,i}| < Mt\delta^* z,$$

where M is a constant, independent of $\delta^* z$.

Thus we have obtained the required estimate of $\Delta I''_{2,i}$. It now remains for us to estimate $\Delta I_{5,i}$. We note first that the estimate (1.4.33) is still valid, since it was not based on the differentiability of f. Thus it is only necessary to estimate

(1.4.66) $$\Delta I'_{5,i} = \int_0^t d\tau \int_0^{y(\tau)} f(\xi,\tau)\delta \frac{\partial}{\partial \xi} E(x(t) + (-1)^i \xi, a^2(t-\tau))\,d\xi$$

for $x(t) \equiv \text{const} \in [0, y(t)]$ and for $x \equiv y(t)$. We have

(1.4.67) $$\Delta I'_{5,i}(x,t,\xi,\tau,0,y(\tau),a|f(\xi,\tau)) \equiv 0 \text{ for } x \equiv \text{const}$$

[9)] The first of the inequalities (1.4.63) follows from (1.4.56).

and

$$|\Delta I'_{5,2}(y(t),t,\xi,\tau,0,y(\tau),a|f(\xi,\tau)| < M\delta^* y, \tag{1.4.68}$$

since $E(y(t)+\xi,a^2(t-\tau))$ is bounded, together with all of its derivatives with respect to ξ and t for $0 \leqq t \leqq T$, $0 \leqq \xi \leqq y(\tau)$, because $y(\tau) \geqq \rho_0 > 0$. Furthermore

$$\begin{aligned}\Delta I'_{5,1} = I' + I'' = &\int_0^t d\tau \int_0^{y(\tau)} f(\xi,t)\delta \frac{\partial}{\partial \xi} E(y(t)-\xi,a^2(t-\tau))d\xi \\ &+ \int_0^t d\tau \int_0^{y(\tau)} [f(\xi,\tau) - f(\xi,t)]\delta \frac{\partial}{\partial \xi} E(y(t)-\xi,a^2(t-\tau))d\xi.\end{aligned} \tag{1.4.69}$$

We now investigate I'. Let

$$0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_m = t$$

be a sequence of interior extrema of $y(t)$ such that $y(t)$ is monotone in each of the intervals (τ_k, τ_{k+1}).

Assume for definiteness that $y(t)$ is increasing on the intervals (τ_{2k}, τ_{2k+1}) and $n = 2m$. Let

$$y_k = y(\tau_k); \quad \tau = \tau_k(\xi) \text{ for } \xi \in (y_k, y_{k+1}). \tag{1.4.70}$$

We find that

$$\begin{aligned}I' = &\sum_{k=0}^{m-1} \left(\int_{\tau_{2k}}^{\tau_{2k+1}} d\tau \int_0^{y_{2k}} f(\xi,t)\delta \frac{\partial E}{\partial \xi} d\xi + \int_{\tau_{2k+1}}^{\tau_{2k+2}} d\tau \int_0^{y_{2k+2}} f(\xi,t)\delta \frac{\partial E}{\partial \xi} d\xi \right) \\ &+ \sum_{k=0}^{m-1} \left(\int_{\tau_{2k}}^{\tau_{2k+1}} d\tau \int_{y_{2k}}^{y(\tau)} f(\xi,t)\delta \frac{\partial E}{\partial \xi} d\xi \right. \\ &\left. + \int_{\tau_{2k+1}}^{\tau_{2k+2}} d\tau \int_{y_{2k+2}}^{y(\tau)} f(\xi,\tau)\delta \frac{\partial E}{\partial \xi} d\xi \right).\end{aligned} \tag{1.4.71}$$

Interchanging the order of integration, we obtain

$$\begin{aligned}I' = &\sum_{k=0}^{m-1} \left(\int_0^{y_{2k}} f(\xi,t)d\xi\delta \int_{\tau_{2k}}^{\tau_{2k+1}} \frac{\partial E}{\partial \xi} d\tau + \int_0^{y_{2k+2}} f(\xi,t)d\xi\delta \int_{\tau_{2k+1}}^{\tau_{2k+2}} \frac{\partial E}{\partial \xi} d\tau \right) \\ &+ \sum_{k=0}^{m-1} \left(\int_{y_{2k}}^{y_{2k+1}} f(\xi,t)d\xi\delta \int_{\tau_{2k}(\xi)}^{\tau_{2k+1}} \frac{\partial E}{\partial \xi} d\tau \right. \\ &\left. + \int_{y_{2k+2}}^{y_{2k+1}} f(\xi,t)d\xi\delta \int_{\tau_{2k+1}}^{\tau_{2k+1}(\xi)} \frac{\partial E}{\partial \xi} d\tau \right).\end{aligned} \tag{1.4.72}$$

Using obvious substitutions in the interior integrals, we see that

$$I' = \sum_{k=0}^{m-1} \frac{1}{a^2\sqrt{\pi}} \left(\int_0^{y_{2k}} f(\xi,t)\,d\xi\delta \int_{\zeta_{2k}}^{\zeta_{2k+1}} \exp(-\lambda^2)\,d\lambda \right.$$

$$\left. + \int_0^{y_{2k+2}} f(\xi,t)\,d\xi\delta \int_{\zeta_{2k+1}}^{\zeta_{2k+2}} \exp(-\lambda^2)\,d\lambda \right)$$

(1.4.73)

$$+ \sum_{k=0}^{m-1} \frac{1}{a^2\sqrt{\pi}} \left(\int_{y_{2k}}^{y_{2k+1}} f(\xi,t)\,d\xi\delta \int_{\zeta_{2k}(\xi)}^{\zeta_{2k+1}} \exp(-\lambda^2)\,d\lambda \right.$$

$$\left. + \int_{y_{2k+2}}^{y_{2k+1}} f(\xi,t)\,d\xi\delta \int_{\zeta_{2k+1}}^{\zeta_{2k+1}(\xi)} \exp(-\lambda^2)\,d\lambda \right) = I_1' + I_2'.$$

Here

$$\zeta_i = \frac{y(t) - \xi}{2a\sqrt{t - \tau_i}}; \quad \zeta_i(\xi) = \frac{y(t) - \xi}{2a\sqrt{t - \tau_i(\xi)}}. \tag{1.4.73*}$$

We note that for $k = m - 1$

$$\frac{y(t) - \xi}{2a\sqrt{t - \tau_{2k+2}}} = \infty; \quad \frac{y^*(t) - \xi}{2a\sqrt{t - \tau_{2k+2}}} = \infty \tag{1.4.74}$$

and that

$$t - \tau_k = O(t) \text{ for } k \neq 2m. \tag{1.4.75}$$

Hence

$$|I_1'| < M\frac{\delta^* y}{\sqrt{t}},$$

or equivalently

$$|I_1'| < M\sqrt{t}\delta^* z. \tag{1.4.76}$$

We now examine I_2'. Suppose that $y(t)$ varies in the expression for I_2', but $\tau_{2k}(\xi)$ and $\tau_{2k+1}(\xi)$ do not. We will temporarily write $\bar{y}(t)$ in place of $y(t)$, indicating by the bar that y is subject to variation. We set $\tau = \tau_{2k}(\xi)$ in the first of the integrals under the summation sign, as an independent variable; we set $\tau = \tau_{2k+1}(\xi)$ in the second; we then find that

$$I_2' = \sum_{k=0}^{m-1} \frac{1}{a^2\sqrt{\pi}} \left(\int_{\tau_{2k}}^{\tau_{2k+1}} f(y(\tau),t)\frac{dy}{d\tau}\,d\tau\delta \int_{\bar{\zeta}(\tau)}^{\zeta_{2k+1}} \exp(-\lambda^2)\,d\lambda \right.$$

(1.4.77)

$$\left. + \int_{\tau_{2k+1}}^{\tau_{2k+2}} f(y(\tau),t)\frac{dy}{d\tau}\,d\tau\delta \int_{\zeta_{2k+1}}^{\bar{\zeta}(\tau)} \exp(-\lambda^2)\,d\lambda \right),$$

where

$$\bar{\zeta}(\tau) = \frac{\bar{y}(t) - y(\tau)}{2a\sqrt{t-\tau}}; \quad \bar{\zeta}_{2k+1} = \frac{\bar{y}(t) - y(\tau)}{2a\sqrt{t-\tau_{2k+1}}}. \tag{1.4.77*}$$

But

$$\left| \delta \int_{\bar{\zeta}(\tau)}^{\zeta_{2k+1}} \exp(-\lambda^2) d\lambda \right| < \frac{\delta^* y}{2a} \left(\frac{1}{\sqrt{t-\tau_{2k+1}}} + \frac{1}{\sqrt{t-\tau}} \right).$$

Hence, as in the estimate of I_1', it follows that

$$|I_2'| < M\sqrt{t}\delta^* z. \tag{1.4.78}$$

Combining the above estimates, we finally obtain

$$|I'| < M\sqrt{t}\delta^* z. \tag{1.4.79}$$

We now examine I''. We have

$$\begin{aligned} I'' = & \int_0^t d\tau \int_0^{y(\tau)} \frac{f(\xi,\tau) - f(\xi,t)}{4a^3\sqrt{\pi}(t-\tau)^{3/2}} \exp\left(-\frac{(y^*(t)-\xi)^2}{4a^2(t-\tau)} \right) \delta y(t) d\xi \\ & - \int_0^t d\tau \int_0^{y(\tau)} \frac{f(\xi,\tau) - f(\xi,t)}{4a^3\sqrt{\pi}(t-\tau)^{3/2}} (y(t)-\xi) d\xi \\ & \cdot \int_{y(t)-\xi}^{y^*(t)-\xi} \frac{\zeta}{2a^2(t-\tau)} \exp\left(-\frac{\zeta^2}{4a^2(t-\tau)} \right) d\zeta = I_1'' + I_2''. \end{aligned} \tag{1.4.80}$$

From (1.4.56) we easily find that

$$|I_1''| < M\sqrt{t}\delta^* y. \tag{1.4.81}$$

Therefore

$$\begin{aligned} I_2'' = & \int_0^t d\tau \int_0^{y(\tau)} \frac{f(\xi,t) - f(\xi,\tau)}{4a^3\sqrt{\pi}(t-\tau)^{3/2}} (y(t)-\xi) d\xi \\ & \cdot \int_0^{\delta y(t)} \frac{\zeta + y(t) - \xi}{2a^2(t-\tau)} \exp\left(-\frac{(\zeta + y(t) - \xi)^2}{4a^2(t-\tau)} \right) d\zeta. \end{aligned} \tag{1.4.82}$$

Reversing the order of integration in the multiple integral, we obtain

$$\begin{aligned} I_2'' = & \int_0^{\delta y(t)} d\zeta \int_0^t d\tau \int_0^{y(\tau)} \frac{f(\xi,t) - f(\xi,\tau)}{8a^5\sqrt{\pi}(t-\tau)^{5/2}} \cdot (y(t)-\xi) \cdot (\zeta + y(t) - \zeta) \\ & \cdot \exp\left(-\frac{(\zeta + y(t) - \xi)^2}{4a^2(t-\tau)} \right) d\xi. \end{aligned} \tag{1.4.83}$$

Again using (1.4.56) and substituting $\zeta + y(t) - \xi = 2a\lambda\sqrt{t-\tau}$, we see that the interior double integral in (1.4.83) is $O(1)$, whence

$$|I_2''| < Mt\delta^* z. \tag{1.4.84}$$

Combining (1.4.84) and (1.4.81), we find that

$$|I''| < Mt\delta^* z, \tag{1.4.85}$$

which, together with (1.4.79) and (1.4.33), yields

$$|\Delta I_{5,i}| < M\sqrt{t}\delta^* z. \tag{1.4.86}$$

Thus the validity of the estimate (1.4.53) for the case (1.4.56) is completely established.

Since the uniform boundedness of the sequences $\{\sqrt{t}\,\partial u_n/\partial t\}, \cdots, \{\sqrt{t}\,dz_n/dt\}$ implies that the aggregate $\{u_n\}, \cdots, \{z_n\}$ uniformly satisfies the conditions (1.4.56), we see that the uniqueness theorem is valid in every class of functions in which existence of the solution is established.

§5. Stability of the solution

The above estimates can be used to prove Theorem 4, which asserts the stability of the solution under small variations of all known quantities appearing in the problem. We first present a more precise preliminary formulation of this theorem.

We will refer to a, l, f, ψ, ϕ, Z and F as "parameters" (or input data) of the problem. Together with $a, l, \cdots, F$ we are given a second system of parameters $\bar{a}, \bar{l}, \cdots, \bar{F}$, having the same differentiability properties and satisfying the conditions arising from (1.1.9)—(1.1.11) on the replacement of $a, \cdots, F$ by $a, \cdots, \bar{F}$. Furthermore let $u, \cdots, y$ and $\bar{u}, \cdots, \bar{y}$ be defined simultaneously for $0 \leqq t \leqq T$, where for $0 \leqq t \leqq T$ and $0 \leqq x \leqq 1$,

$$\begin{gathered} |u|;\quad |w|;\quad |q|;\quad |v|;\quad |z| < N; \\ |\bar{u}|;\quad |\bar{w}|;\quad |\bar{q}|;\quad |\bar{v}|;\quad |\bar{z}| < N; \\ 0 \leqq y(t) \leqq 1;\quad 0 \leqq \bar{y}(t) \leqq 1. \end{gathered} \tag{1.5.1}$$

We then assert

THEOREM 4 (STABILITY OF THE SOLUTION). *For any $\epsilon > 0$ it is possible to find some $\eta > 0$ such that in an interval $0 \leqq t \leqq T_1$ (with $T_1 \leqq T$ independent of ϵ) the inequalities*

$$\Delta u < \epsilon;\quad \Delta q < \epsilon;\quad \Delta w < \epsilon;\quad \Delta z < \epsilon;\quad \Delta y < \epsilon \tag{1.5.2}$$

are fulfilled, as soon as the absolute value of the difference of a^2 *and* $\bar{a}^2$, l *and* $\bar{l}$, F, f, Z, ψ *and* ϕ *and respectively* $\bar{F}$, $\bar{f}$, $\bar{Z}$, $\bar{\psi}$ *and* $\bar{\phi}$, *and all of their partial derivatives of first order, taken for coincident values of the argument satisfying inequalities* (1.5.1), *are not greater than* η. *Here*

$$\Delta u = \max_{\substack{0\le x\le L;\\ 0\le t\le T_1}} |u(x,t)-\bar{u}(x,t)|; \quad \Delta q = \max_{\substack{0\le x\le L;\\ 0\le t\le T_1}} |q(x,t)-\bar{q}(x,t)|;$$

$$\Delta z = \max_{0\le t\le T_1} |z(t)-\bar{z}(t)|; \quad \Delta y = \max_{0\le t\le T_1} |y(t)-\bar{y}(t)|; \tag{1.5.3}$$

$$\Delta v = \max_{0\le t\le T_1} |v(t)-\bar{v}(t)|; \quad \Delta w = \max_{0\le t\le T_1} |w(t)-\bar{w}(t)|$$

and

$$L = L(t) = \min(y(t), \bar{y}(t)). \tag{1.5.3*}$$

Assume first that

$$\bar{a} = a; \quad \bar{l} = l. \tag{1.5.4}$$

Then to estimate $\Delta u, \cdots, \Delta y$, one can use the estimates (1.4.39) and (1.4.41) of the functional $I_{i,j}$.[10]

Let $f(x,t,\alpha_1,\cdots,\alpha_n)$ and $\bar{f}(x,t,\alpha_1,\cdots,\alpha_n)$ be defined and differentiable for $0 \le x \le 1$, $0 \le t \le T_1$ and $|\alpha_k| < N$ $(k = 1,\cdots,n)$. Set

$$R_f = \max\max \left\{ |f-\bar{f}|; \left|\frac{\partial f}{\partial x}-\frac{\partial \bar{f}}{\partial x}\right|; \left|\frac{\partial f}{\partial t}-\frac{\partial \bar{f}}{\partial t}\right|; \left|\frac{\partial f}{\partial \alpha_k}-\frac{\partial \bar{f}}{\partial \alpha_k}\right| \right\}; \tag{1.5.5}$$

$$k = 1,2,\cdots,n; \quad 0 \le x \le 1; \quad 0 \le t \le T; \quad |\alpha_k| < N.$$

Furthermore, let

$$|f|;\ |\bar{f}|;\ \left|\frac{\partial f}{\partial x}\right|;\ \left|\frac{\partial \bar{f}}{\partial x}\right|;\ \left|\frac{\partial f}{\partial t}\right|;\ \left|\frac{\partial \bar{f}}{\partial t}\right|;\ \left|\frac{\partial f}{\partial \alpha_k}\right|;\ \left|\frac{\partial \bar{f}}{\partial \alpha_k}\right| \le K; \tag{1.5.6}$$

$$k = 1,2,\cdots,n,$$

where K is a constant independent of R_f. We have

$$|f(x,t,u,w,q,v,z,y) - \bar{f}(x,t,\bar{u},\bar{w},\bar{q},\bar{v},\bar{z},\bar{y})| \tag{1.5.7}$$
$$\le R_f + K(\Delta u + \Delta w + \Delta q + \Delta v + \Delta z + \Delta y).$$

We now examine the variations of the functionals U, W, Q, V and Z induced by variations of the parameters of the problem and consequent

[10] By virtue of the result obtained at the end of §4 this estimate remains valid for a nondifferentiable solution satisfying (1.4.5).

variations of u, w, q, v, z and y. We use the notation defined above. By (1.4.42), (1.3.4), (1.3.5) and (1.5.7) we obtain, as in the derivation of (1.4.43),

$$
\begin{aligned}
&\delta^* U < K\sqrt{t}(\Delta u + \Delta w + \Delta q + \Delta v + \Delta z) + K^* R;\\
&\delta^* W < K\sqrt{t}(\Delta u + \Delta w + \Delta q + \Delta v + \Delta z) + K^* R;\\
&\delta^* Q < K\sqrt{t}(\Delta u + \Delta q + \Delta z) + \bar{K}(\Delta w + \Delta v) + K^* R;\\
&\delta^* V < K\sqrt{t}(\Delta u + \Delta w + \Delta q + \Delta v + \Delta z) + K^* R;\\
&\delta^* Z < \bar{K}\sqrt{t}\Delta z + \bar{K}\Delta v + K^* R.
\end{aligned}
\tag{1.5.8}
$$

Here K, $\bar{K}$, K^* are constants which are independent of $\Delta u, \cdots, \Delta z$ and R, and

$$
R = \max\{R_f, R_F, R_Z, R_\psi, R_\phi\}. \tag{1.5.9}
$$

Since $u, \cdots, y$ and respectively $\bar{u}, \cdots, \bar{y}$ are the solutions to the problems corresponding to comparable values of the parameters, it is possible to replace $\delta^* U, \cdots, \delta^* Z$ by $\Delta u, \cdots, \Delta z$ in (1.5.8). As a result we find that

$$
\Delta u;\quad \Delta w;\quad \Delta q;\quad \Delta v;\quad \Delta z;\quad \Delta y < LR, \tag{1.5.10}
$$

where L is a suitably chosen constant, growing with t but independent of $\Delta u, \cdots, \Delta y$. This completes the proof of our assertion, assuming the inequalities (1.5.4).

Suppose now that (1.5.4) is not true. We set

$$
\begin{aligned}
&x^* = \frac{l}{\bar{l}}x;\quad \xi^* = \frac{l}{\bar{l}}\xi;\quad t^* = \frac{\bar{a}^2 l^2}{a^2 \bar{l}^2}t;\quad \tau^* = \frac{\bar{a}^2 l^2}{a^2 \bar{l}^2}\tau;\\
&y^*(t^*) = \frac{l}{\bar{l}}\bar{y}(t);\quad z^*(t^*) = \frac{\bar{l}a^2}{l\bar{a}^2}\bar{z}(t);\\
&u^*(x^*, t^*) = \bar{u}(x, t);\quad w^*(t^*) = \bar{w}(t);\\
&q^*(x^*, t^*) = \frac{\bar{l}}{l}\bar{q}(x, t);\quad v^*(t^*) = \frac{\bar{l}}{l}\bar{v}(t).
\end{aligned}
\tag{1.5.11}
$$

Furthermore, set

$$
\begin{aligned}
&F^*(x^*, t^*, u^*, \cdots, y^*) = \frac{a^2 \bar{l}^2}{\bar{a}^2 l^2}\bar{F}(x, t, \bar{u}, \cdots, \bar{y});\\
&f^*(t^*, w^*) = \frac{\bar{l}}{l}\bar{f}(t, \bar{w});\quad \psi^*(x^*) = \bar{\psi}(x);\quad \phi^*(x^*) = \frac{\bar{l}}{l}\bar{\phi}(x);\\
&Z^*(t^*, \psi^*, v^*, y^*) = \bar{Z}(t, \bar{\psi}, \bar{v}, \bar{y})\frac{\bar{l}a^2}{l\bar{a}^2}.
\end{aligned}
\tag{1.5.12}
$$

We then arrive at the equations

$$
\begin{aligned}
&u^*(x^*,t^*) = U(x^*,t^*,u^*,\cdots,y^*);\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&y^*(t^*) = Y(t^*,\cdots,z^*),
\end{aligned}
\tag{1.5.13}
$$

where, as above, $U,\cdots,Y$ are operators defined by the equations $(1.1.12_1)$—$(1.1.12_6)$. We now have

$$
\begin{aligned}
&\Delta u = \max|u(x,t) - u^*(x^*,t^*)|;\\
&\Delta w = \max|w(t) - w^*(t^*)|;\\
&\Delta q = \max\left|q(x,t) - \frac{l}{\bar{l}}q^*(x^*,t^*)\right|;\\
&\Delta v = \max\left|v(t) - \frac{l}{\bar{l}}v^*(t^*)\right|;\\
&\Delta z = \max\left|z(t) - \frac{l\bar{a}^2}{\bar{l}a^2}z^*(t^*)\right|;\\
&\Delta y = \max\left|y(t) - \frac{\bar{l}}{l}y^*(t^*)\right|.
\end{aligned}
\tag{1.5.14}
$$

By (1.5.12) $u,\cdots,y$ and $u^*,\cdots,y^*$ correspond to equal values of a and l. Consequently the values

$$
\begin{aligned}
&\Delta^* u = \max|u(x,t) - u^*(x,t)|;\\
&\cdots\cdots\cdots\cdots\cdots\\
&\Delta^* y = \max|y(t) - y^*(t)|
\end{aligned}
\tag{1.5.15}
$$

can be estimated by (1.5.10). But the differences

$$|\Delta u - \Delta^* u|,\cdots,|\Delta y - \Delta^* y|$$

tend to zero together with

$$\alpha = |a - \bar{a}| + |l - \bar{l}|$$

by virtue of the continuity of $u,\cdots,y$ and $\bar{u},\cdots,\bar{y}$ with respect to x and t on the one hand, and the equations

$$\lim_{\alpha\to 0} x^* = x;\quad \lim_{\alpha\to 0} t^* = t,$$

on the other. But this implies the validity of our assertion.

The stability of the solution was proved, generally speaking, in a small time interval $(0, T_1)$, rather than in an interval $(0, T)$ in which solutions of the perturbed and unperturbed problems are simultaneously

defined. However, since T_1 is not dependent on ϵ, but only on the bounds of the "parameters" f, F, ψ, ϕ, Z and their derivatives (and moreover on the estimates of the solutions $u, w, \cdots, y$ and $\bar{u}, \bar{w}, \cdots, \bar{y}$), it is obvious that the conclusions of the smallness of $\Delta u, \cdots, \Delta y$ can be extended to the entire interval $(0, T)$.

In fact, let T_1 be the greatest length of a time interval on which the inequalities

$$\begin{aligned} &|\bar{a} - a| < \eta; \quad |l - \bar{l}| < \eta; \quad R_F^{t_0, T_1} < \eta; \\ &T_z^{t_0, T_1} < \eta; \quad R_\psi < \eta; \quad R_{u(x,t_0)} < \eta \end{aligned} \tag{1.5.16}$$

imply the inequalities

$$\Delta_{t_0} u < \epsilon; \quad \Delta_{t_0} w < \epsilon; \quad \Delta_{t_0} q < \epsilon; \quad \Delta_{t_0} v < \epsilon; \quad \Delta_{t_0} z < \epsilon; \quad \Delta_{t_0} y < \epsilon. \tag{1.5.17}$$

Here $R_f^{t_0, T_1}$ is defined exactly as R_f in (1.5.5) but on the interval $t_0 \leqq t \leqq T_1 + T_0$ instead of on the interval $0 \leqq t \leqq T$, and

$$\begin{aligned} \Delta_{t_0} u &= \max_{\substack{t_0 \leqq t \leqq t_0 + T_1; \\ 0 \leqq x \leqq \min(y(t)\bar{y}(t))}} |u(x,t) - \bar{u}(x,t)|; \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \Delta_{t_0} y &= \max_{t_0 \leqq t \leqq t_0 + T_1} |y(t) - \bar{y}(t)|. \end{aligned} \tag{1.5.18}$$

As we indicated above, T_1 is not dependent on ϵ, η and t_0. Let $\epsilon(\eta)$ be a monotonically increasing function such that the inequality (1.5.2) is valid when ϵ is replaced by $\epsilon(\eta)$. Let $\eta(\epsilon)$ be the inverse of this function. Further, let n be a number such that

$$n - 1 < T/T_1 \leqq n. \tag{1.5.19}$$

Set $t_k = kT_1$ $(k = 0, \cdots, n-1)$. Choose $\epsilon > 0$ arbitrarily small and define the sequence

$$\epsilon_{n-1} = \epsilon; \quad \epsilon_{n-k} = \eta(\epsilon_{n-k+1}) \quad (k = 1, 2, \cdots, n-1); \quad \eta = \epsilon_0. \tag{1.5.20}$$

In this case the inequalities

$$\begin{aligned} &|a - \bar{a}| < \eta; \quad |l - \bar{l}| < \eta; \quad R_f^{0, T_1} < \eta; \quad R_F^{0, T_1} < \eta; \\ &R_z^{0, T_1} < \eta; \quad R_\psi < \eta; \quad R_{u(x,0)=\phi} < \eta \end{aligned} \tag{1.5.21}$$

guarantee the validity of (1.5.17) on each of the intervals (t_k, t_{k+1}) $(k = 0, \cdots, n-1)$, and at the same time on the entire interval $(0, T)$. Hence our assertion is proved.

§6. Further observations

A. **Application of the method of polygonal approximations of Cauchy-Lipschitz type** [176]. From the proof of convergence of the Picard iteration scheme and from the stability of the solution with respect to every parameter of the problem, we see that it is clearly possible to carry out the following iteration-differencing process.

Choose some $T > 0$ sufficiently small and define the sequence

$$\begin{aligned} &t_0 = 0 < t_1 < \cdots < t_n = T; \\ &x_0 = 0 < x_1 < \cdots < x_{m_n - 1} < x_{m_n} = l < \cdots < x_n = 1. \end{aligned} \tag{1.6.1}$$

Consider the functions $u_{n,k}(x,t)$, $q_{n,k}(x,t)$, $w_{n,k}(t)$, $v_{n,k}(t)$, $z_{n,k}(t)$, $y_{n,k}(t)$, $\bar{u}_{n,k}(x,t)$, $\bar{q}_{n,k}(x,t)$, $\bar{w}_{n,k}(t)$, $\bar{v}_{n,k}(t)$, $\bar{z}_{n,k}(t)$, $\bar{y}_{n,k}(t)$, $u_n(x,t)$, $q_n(x,t)$, $w_n(t)$, $v_n(t)$, $z_n(t)$, $y_n(t)$, determined by the following conditions A_1—A_8.

A_1. $u_{n,k}(x,t)$, $q_{n,k}(x,t)$, $w_{n,k}(t)$, $v_{n,k}(t)$, $z_{n,k}(t)$ and $y_{n,k}(t)$ are continuous, together with their derivatives of first order, everywhere within the domain $0 \leqq x \leqq y_{n,k}(t)$; $0 \leqq t \leqq t_n$.

A_2.
$$\begin{aligned} &u_{n,k}(x_j, 0) = \phi(x_j); \quad q_{n,k}(x_j, 0) = \dot{\phi}(x_j); \\ &w_{n,k}(0) = \phi(0); \quad v_{n,k}(0) = \dot{\phi}(l); \\ &z_{n,k}(0) = Z(0, \psi(l), \dot{\phi}(l), l); \quad y_{n,k}(0) = l. \end{aligned} \tag{1.6.2}$$

A_3. If u_n, q_n, w_n, v_n, z_n and y_n are defined for $0 \leqq t \leqq t_i$, and $u_{n,k}$, $q_{n,k}$, $w_{n,k}$, $v_{n,k}$, $z_{n,k}$ and $y_{n,k}$ for $0 \leqq t \leqq t_{i+1}$, then

$$\begin{aligned} &\bar{u}_{n,k} = \begin{cases} u_n & \text{for } 0 \leqq t \leqq t_i; \\ u_{n,k} & \text{for } t_i < t \leqq t_{i+1}; \end{cases} \qquad \bar{w}_{n,k} = \begin{cases} w_n & \text{for } 0 \leqq t \leqq t_i; \\ w_{n,k} & \text{for } t_i < t \leqq t_{i+1}; \end{cases} \\ &\bar{q}_{n,k} = \begin{cases} q_n & \text{for } 0 \leqq t \leqq t_i; \\ q_{n,k} & \text{for } t_i < t \leqq t_{i+1}; \end{cases} \qquad \bar{v}_{n,k} = \begin{cases} v_n & \text{for } 0 \leqq t \leqq t_i; \\ v_{n,k} & \text{for } t_i < t \leqq t_{i+1}; \end{cases} \\ &\bar{z}_{n,k} = \begin{cases} z_n & \text{for } 0 \leqq t \leqq t_i; \\ z_{n,k} & \text{for } t_i < t \leqq t_{i+1}; \end{cases} \qquad \bar{y}_{n,k}(t) = l + \int_0^t \bar{z}_{n,k}(\tau)\, d\tau. \end{aligned} \tag{1.6.3}$$

A_4.
$$\begin{aligned} &u_n = \lim_{k \to \infty} u_{n,k}; \quad q_n = \lim_{k \to \infty} q_{n,k}; \quad w_n = \lim_{k \to \infty} w_{n,k}; \\ &v_n = \lim_{k \to \infty} v_{n,k}; \quad z_n = \lim_{k \to \infty} z_{n,k}; \quad y_n = \lim_{k \to \infty} y_{n,k}. \end{aligned} \tag{1.6.4}$$

A_5. If $u_n, \cdots, y_n$ are defined for $0 \leqq t \leqq t_j$, then

$$u_{n,0}(x_i, t_{j+1}) = u_n(x_i, t_j), \cdots, y_{n,0}(t_{j+1}) = y_n(t_j). \tag{1.6.5}$$

A_6. Let $\bar{u}_{n,k}, \cdots, \bar{z}_{n,k}, \bar{y}_{n,k}$ and their derivatives with respect to x and t be defined for $0 \leqq t \leqq t_j$. Furthermore, suppose that $W_{n,k}, \cdots, Z_{n,k}$ coincide with the functionals $W_n, \cdots, Z_n$ defined in (1.1.16), if $w_m, \cdots, y_m$ $(m = n-1;\ n)$ are replaced by $\bar{w}_{n,m}, \cdots, \bar{y}_{n,m}$ $(m = k-1;\ k)$. Now let $W^*_{n,k}, \cdots, Z^*_{n,k}$ be the values of the derivatives of the functionals $W_{n,k}, \cdots, Z_{n,k}$ defined by equations (1.3.34), (1.3.35), (1.3.28), (1.3.29) and (1.3.30). Similarly, let $Q^{**}_{n,k}$ be the derivative of $Q_{n,k}$ with respect to x. Then

$$
\begin{aligned}
w_{n,k+1}(t_j) &= W_{n,k+1}(t_j, \cdots);\\
u_{n,k+1}(x_i, t_j) &= U_{n,k+1}(x_i, t_j, \cdots);\\
&\cdots\cdots\cdots\cdots\cdots\\
z_{n,k+1}(t_j) &= Z_{n,k+1}(t_j, \cdots);
\end{aligned}
\tag{1.6.6}
$$

$$
\begin{aligned}
\frac{dw_{n,k+1}(t_j)}{dt} &= W^*_{n,k+1}(t_j, \cdots);\\
&\cdots\cdots\cdots\cdots\cdots\\
\frac{dz_{n,k+1}(t_j)}{dt} &= Z^*_{n,k+1}(t_j, \cdots);
\end{aligned}
\tag{1.6.7}
$$

$$
\frac{\partial}{\partial x} q_{n,k+1}(x_i, t_j) = Q^{**}_{n,k+1}(x_i, t_j, \cdots). \tag{1.6.8}
$$

A_7. Within every rectangle $x_i < x < x_{i+1},\ t_{j-1} < t < t_j$, the functions $w_{n,k+1}, \cdots, z_{n,k+1}$ are defined by means of some interpolation formula, guaranteeing continuity of these functions together with their derivatives with respect to x and t in the closure of these rectangles, and assuming values of these functions and their derivatives with respect to x and t only at the vertices

$$
(x_i, t_j);\quad (x_{i+1}, t_j);\quad (x_i, t_{j-1});\quad (x_{i+1}, t_{j-1}).
$$

A_8. For arbitrary fixed t

$$
y_{n,k}(t) = l + \int_0^t z_{n,k}(\tau)\, d\tau. \tag{1.6.9}
$$

Now let $n \to \infty$ and simultaneously let $t_{k+1} - t_k,\ x_{i+1} - x_i$ tend to zero for all i, k. In this case it is possible to prove that $u_{n,k}, \cdots, z_{n,k}$ converge to the values of the solution of the problem, if $T > 0$ is sufficiently small.

The proof of convergence of the finite-difference process can be given by means of estimates already obtained and similar ones. For

this reason we will not dwell on this point. We note only that from the proof of stability of the solution and from the choice of an interpolation formula of suitable accuracy we can obtain a scheme for a mesh size which is not too small with respect to time. It is only necessary that close to $t = 0$ this mesh be finer, since for $t \to 0$ we have $\dot{q} = O(t^{-1/2})$, $\dot{z} = O(t^{-1/2})$ and $\dot{v} = O(t^{-1/2})$. As t increases from $t = 0$ the mesh length in time can be enlarged.

B. **Construction of the solution in the large.** The existence of a solution was proved in a sufficiently small time interval. The problem of extending the solution to an arbitrary preassigned time interval cannot be investigated without an additional condition of boundedness on the input data functions. In the classical two-phase Stefan problem on an interval, this problem is the subject of special investigations in the following chapter. Here we will examine the simplest case, which can be solved without any difficulty. With this aim we show that the time interval in which the solution of the problem exists, which by virtue of Theorem 3 depends on the estimates (1.1.8), (1.1.9), (1.1.10) and the magnitude of $\rho = \min_{0<t<T} y(t)$ tends to zero with ρ and N^{-1}. Here N is the maximum of the numbers M_i, $N_{i,j}$ appearing in (1.1.9) and (1.1.10).

Suppose that F, f, ψ, ϕ and Z are defined for $x \geqq 0$, $t \geqq 0$ and for $-\infty < u, q, v, z, y < \infty$; we also assume that the estimates (1.1.9) and (1.1.10) are valid for any values $u, q, \cdots, z, y, x$, and t. Suppose further that Z remains nonnegative for every value of its argument, so that y is nondecreasing everywhere in its domain of definition. In this case $\rho \geqq l$ is nondecreasing together with T. This shows that if the solution to the problem is defined on the interval $(0, T_0)$, then it can be determined on the interval (T_0, T_1), where $\Delta T = T_1 - T_0$ is dependent only on the magnitude of u and its first three derivatives with respect to x at the time $t = T_0$. Let there be given an arbitrary $T^* > 0$ and the solution to the problem constructed for the time $t < T^*$. As is seen from estimates (1.3.20), (1.3.21), together with the equations (1.3.25), (1.3.26), (1.3.31) and estimates (1.3.38)—(1.3.42), there exists a constant $L > 0$, not depending on t, such that for every $x \in [0, y(t)]$

$$|u(x,t)| < L(t+1); \quad \left|\frac{\partial u}{\partial x}\right| < L(t+1); \quad \left|\frac{\partial^2 u}{\partial x^2}\right| < L(t+1);$$
$$\left|\frac{\partial^2 u}{\partial x \partial t}\right| < L\left(t + \frac{1}{\sqrt{t}}\right). \tag{1.6.10}$$

This shows that for given $T^* > 0$ the a priori estimates of u, $\partial u/\partial x$, $\partial^2 u/\partial x^2$ and $\partial^3 u/\partial x^3$ remain valid; but these depend only on T^* and not on the time t, for which a solution is already constructed. Hence the length of the time interval in which the solution was determined above is not dependent on the value of T_0. Consequently, if the iteration process of Theorem 3 is applied to finding the solution on any time interval in which the process is known to converge, then the length of the interval will be independent of the length of the interval in which the solution has already been defined; this shows that every interval $(0, T^*)$ can be attained by a finite number of steps.[11]

C. **Multi-phase problems.** The theory we have developed is applicable to a multi-phase problem involving an arbitrary number of phases, with the limitation that the domain of existence of each phase does not degenerate to a point and that at the boundary between layers we have the usual jump conditions. Assume that the input data of the problem is subject to conditions analogous to those formulated for a single phase problem.

We will briefly sketch the method for reducing the problem to a system of integral equations. Given $L_i = \text{const}$ $(i = 0, \cdots, n)$, such that

$$L_0 < L_1 < \cdots < L_n, \tag{1.6.11}$$

we seek $u_{i,j}(x,t), y_{i,j}(t)$ $(i = 1, \cdots, n;\ j = 0, \cdots, m_i)$ such that for $0 \leqq t \leqq T$

$$\begin{gathered} y_{i,0}(t) \equiv L_{i-1};\quad y_{i,m_i}(t) \equiv L_i;\quad L_{i-1} < y_{i,j}(t) < L_i; \\ y_{i,j}(t) < y_{i,j+1}(t);\quad i = 1,2,\cdots,n;\quad j = 1,2,\cdots,m_i - 1, \end{gathered} \tag{1.6.12}$$

and $u_{i,j}(x,t)$ and $\partial u_{i,j}(x,t)/\partial x$ are continuous in the closure of the parabolic domain

$$D_{i,j} = \{ y_{i,j-1}(t) < x < y_{i,j}(t);\quad 0 < t \leqq T \};\quad \begin{matrix} i = 1,2,\cdots,n, \\ j = 1,2,\cdots,m_i, \end{matrix}$$

---

[11] Obviously the ease with which we obtained this result depends on the extremely rigid limitations on $f, \cdots, z$. We note that the most stringent is the requirement that $y(t)$ be monotonically increasing. If $y(t)$ approaches the line $x = 0$ arbitrarily closely, then the value of the gradient of u can become arbitrarily large and therefore our reasoning becomes invalid.

With the positivity of the lower bound of y, and its nonmonotonic character, it is possible to show that the length of the step on which the solution is defined diminishes together with the index of the step, and moreover the length of the resulting interval of definition of the solution tends to a definite limit, while the corresponding value of $y(t)$ oscillates.

and satisfy within $D_{i,j}$ the equations

$$L_{a_{i,j}} u_{i,j} + F_{i,j}\left(x; t; u_{i,j}; \frac{\partial u_{i,j}}{\partial x}; \dot{y}_{i,j-1}; y_{i,j-1}; \dot{y}_{i,j}; y_{i,j}\right) = 0, \tag{1.6.13}$$

the initial conditions

$$u_{i,j}(x,0) = \phi_{i,j}(x); \quad y_{i,j}(0) = l_{i,j}; \quad l_{i,j} \neq l_{i,j+1}, \tag{1.6.14}$$

the boundary conditions

$$\begin{aligned} &u_{i,j}(y_{i,j-1}(t),t) = \psi_{i,j-1}(y_{i,j-1}(t)); \\ &u_{i,j}(y_{i,j}(t),t) = \psi_{i,j}(y_{i,j}(t)); \\ &i = 1,2,\cdots,n; \quad j = 2,\cdots,m_i - 1 \end{aligned} \tag{1.6.15}$$

and

$$\frac{\partial}{\partial x} u_{1,1}(L_0,t) = f_{1,1}(t,u_{1,1})\big|_{x=L_0}; \quad \frac{\partial}{\partial x} u_{n,m_n}(L_n,t) = f_{n,m_n}(t,u_{n,m_n})\big|_{x=L_n} \tag{1.6.16}$$

or

$$u_{1,1}(L_0,t) = f_{1,1}\left[t, \frac{\partial}{\partial x} u_{1,1}\right]_{x=L_0}; \quad u_{n,m_n}(L_n,t) = f_{n,m_n}(t,u_{n,m_n})\big|_{x=L_n}, \tag{1.6.16*}$$

the general Stefan conditions

$$\frac{dy_{i,j}}{dt} = Z_{i,j}\left(t, u_{i,j}, \frac{\partial u_{i,j}}{\partial x}, u_{i,j+1}, \frac{\partial u_{i,j+1}}{\partial x}, y_{i,j}\right) \tag{1.6.17}$$

and the conjugacy conditions

$$\left.\begin{aligned} &u_{i,m_i} = u_{i+1,1}; \\ &k_{i,m_i} \frac{\partial}{\partial x} u_{i,m_i} = k_{i+1,1} \frac{\partial}{\partial x} u_{i+1,1} \end{aligned}\right\} \begin{aligned} &x = L_i \\ &i = 1,2,\cdots,n-1. \end{aligned} \tag{1.6.18}$$

Here $a_{i,j} = \text{const} > 0$; $k_i = \text{const} > 0$ $(i = 1,\cdots,n;\ j = 0,\cdots,m_i)$.

The reduction to an integral equation is done in the following way. Suppose for definiteness that (1.6.16*) holds at the boundary $x = L_0$ and (1.6.16) holds at $x = L_n$. In this case we will express $U_{1,1}$ by means of the fundamental formula, using the Green's function of the first boundary problem for the halfspace $x > L_0$; u_{n,m_n} using the Green's function of the second boundary problem for the halfspace $x < L_n$; and $u_{i,j}$ $(j = 2,\cdots,m_i - 1)$ using $E(x-\xi,\ \alpha_{i,j}^2 t)$. Finally, to eliminate the jump condition (1.6.18) from the calculation for determining $u_{i+1,1}$ and u_{i,m_n} $(i = 1,\cdots,n-1)$ we use the fundamental solution to the heat

conduction equation for a double layer and its conjugate function constructed in Supplement 2, §2. In this case, when in one of the layers, say the kth, the system is one-phased (such that $m_k = 1$), the jump condition (1.6.18) is automatically realized if u_{k-1}, u_k and u_{k+1} are expressed by the fundamental solution to the triple-layer problem. Analogously, in the case of contact of m single phases it is necessary to use the fundamental solution to the heat equation for an $(m+1)$-layered space. Thus in principle the jump conditions (1.6.18) do not create any additional difficulties. However, the presence of a contact between the single phase layers introduces technical difficulties due to the awkwardness of the expressions of the corresponding fundamental solutions. In view of this fact, there is an advantage in introducing the value $u_{i,j}(x,t)$ at the boundary $x = L_k$ as an additional unknown. At the present time we limit ourselves to this observation, referring to the details in Chapters V and VI, where this problem is investigated under concrete conditions.

It is obvious that every assertion above is relevant to the case when in each of the domains $D_{i,j}$ we seek not one function but a system of functions, provided that the boundary conditions satisfied by the functions of this system at the boundaries $x = y_{i,j}(t)$ are independent of one another. For example, our analysis can be applied to the problem of a solid body moving at high speed in a stream in the nonsimilarity formulation (see Part 1, Chapter IV, §2). Conversely, the problem of the dynamics of crystallization of a binary melt with eutectics (see Part 1, Chapter II, §3) requires a separate investigation, since the boundary values of temperature and concentration can be shown to be incompatible. In exactly the same way problems of viscous plastic flow, the theory of which was examined in Part 1, Chapter IV, §3, require special investigations. Finally, every problem in which we have degeneration of the initial domain of existence of one of the phases or a lack of continuous agreement in boundary and initial conditions requires special consideration.

CHAPTER II

THE CLASSICAL TWO-PHASE STEFAN PROBLEM. THE CASE OF A TWO-PHASE INITIAL STATE WITH CONTINUOUS AGREEMENT OF BOUNDARY AND INITIAL CONDITIONS [131], [143]

§1. Statement of the problem

Suppose that numbers l_i and functions $f_i(t)$ and $\phi_i(x)$ $(i = 1,2)$ are given such that $l_1 \leqq l < l_2$, the functions $f_i(t)$ $(i = 1,2)$ are defined and twice continuously differentiable for $0 \leqq t < \infty$, where

$$\begin{aligned} &f_1(t) < 0; \quad f_2(t) > 0, \quad \text{for } t > 0; \\ &\lim_{t \to \infty} |f_i(t)| = T_i > 0 \quad (i = 1,2), \end{aligned} \tag{2.1.1}$$

and $\phi_1(x)$ and $\phi_2(x)$ are defined and three times continuously differentiable for $l_1 \leqq x \leqq l$ and $l \leqq x \leqq l_2$, respectively, where

$$\phi_1(x) < 0 \text{ for } l_1 \leqq x < l; \quad \phi_2(x) > 0 \text{ for } l < x \leqq l_2. \tag{2.1.2}$$

In the case $l = l_1$ we will write $\phi(x)$ in place of $\phi_2(x)$.

Furthermore, let $D_i(y, t_0, l_i, l)$ $(i = 1,2)$ be parabolic regions

$$\begin{aligned} D_1(y, t_0, l_1, l) &= \{l_1 < x < y(\tau);\ t_0 < \tau \leqq t < \infty;\ y(t_0) = l\}; \\ D_2(y, t_0, l_2, l) &= \{y(\tau) < x < l_2;\ t_0 < \tau \leqq t < \infty;\ y(t_0) = l\}. \end{aligned} \tag{2.1.3}$$

Where it is not ambiguous, we will write D_i instead of $D_i(y, t_0, l_i, l)$. As above, we denote the closure of D_i by $\overline{D}_i$.

In this and subsequent chapters we will investigate the following problems.

Problems A_1 and A_3. (The first and third two-phase Stefan problems on an interval with single phase initial conditions.)

Find $u_i(x,t)$ $(i = 1,2)$ and $y(t)$ such that:

a—1: $u_i(x,t)$ is defined and uniformly bounded in $D_i(y, 0, l_i, l_1)$.

a—2: u_i and $\partial u_i / \partial x$ are continuous in $\overline{D}_i$ everywhere, with the possible exception of the points $(l_i, 0)$ $(i = 1,2)$.

a—3:

$$L_i(u_i) = 0, \quad (x,t) \in D_i \qquad (i = 1,2); \tag{2.1.4$_1$}$$

a—4:

$$u_2(x,0) = \phi(x), \quad l_1 < x < l_2; \tag{2.1.4$_2$}$$

a—5:

$$u_1(l_1,t) = f_1(t) \text{ for } \left(\frac{\partial}{\partial x} - h\right) u_1(l_1,t) = -hf_1(t), \quad t > 0; \tag{2.1.4$_3$}$$

a—6:

$$u_2(l_2,t) = f_2(t), \quad t > 0; \tag{2.1.4$_4$}$$

a—7:

$$u_i(y(t),t) = 0 \quad (i = 1,2), \quad t > 0; \tag{2.1.4$_5$}$$

a—8:

$$\dot{y}(t) = \frac{\partial}{\partial x} u_1(y(t),t) - \frac{\partial}{\partial x} u_2(y(t),t), \quad t > 0; \tag{2.1.4$_6$}$$

a—9:

$$y(0) = l_1. \tag{2.1.4$_7$}$$

Conditions (a—1)—(a—3) assert that u_1 is parabolic and u_2 is a-parabolic in D_1 and D_2 respectively. (See the Introduction, §2.) The first of the conditions a—5 is applied to problem A_1, and the second to problem A_3.

Problems B_1 and B_3. (First and third two-phase Stefan problems on an interval for two-phase initial conditions) are posed analogously to problems A_1 and A_3, but with the replacement of the domain $D_i(y,0,l_i,l_1)$ by the domain $D_i(y,t_0,l_i,l)$, $t_0 \geqq 0$, $l_1 < l < l_2$, and the conditions (2.1.4$_2$) and (2.1.4$_7$) by the conditions

$$\begin{aligned} u_1(x,0) &= \phi_1(x), \quad l_1 < x < l; \\ u_2(x,0) &= \phi_2(x), \quad l < x < l_2. \end{aligned} \tag{2.1.4$_2^*$}$$

$$y(0) = l, \quad l_1 < l < l_2. \tag{2.1.4$_7^*$}$$

Problem C (the Cauchy-Stefan problem). Let $l_1 = -\infty$, $l_2 = \infty$, $l = 0$. We seek $u_i(x,t)$ $(i = 1,2)$ and $y(t)$ such that:

b—1: u_i is defined in D_i and uniformly bounded in every bounded part of D_i.

b—2: u_i and $\partial u_i/\partial x$ are continuous in $\overline{D}_i$ everywhere, except possibly at the point $(0,0)$.

$$(2.1.5)\quad \begin{cases} \text{b—3:} \\ \quad u_i = \phi_i \text{ for } t = 0 \quad (-\infty < x < 0,\ i = 1;\ 0 < x < \infty,\ i = 2); \\ \text{b—4:} \\ \quad u_i = 0 \text{ for } x = y(t),\ t > 0; \\ \text{b—5:} \\ \quad \dot{y}(t) = \dfrac{\partial}{\partial x} u_1(y(t),t) - \dfrac{\partial}{\partial x} u_2(y(t),t),\ t > 0;\ y(0) = 0. \end{cases}$$

In the thermal interpretation this problem coincides with the classical Stefan problem for a two-phase heat-conducting medium when the phase transfer occurs isothermally. In this formulation u_1 and u_2 denote the temperatures of the solid and liquid phases respectively, when these temperatures are measured in scales proportional to the coefficients of heat conduction of the corresponding phases and the melting temperature is set equal to a reference temperature zero.

Additional conditions which will be adjoined to problems A_i, B_i and C $(i = 1,3)$ in their individual investigations are formulated below. In the following we will always use the notation

$$(2.1.6)\qquad \frac{\partial}{\partial x} u_i(y(t),t) \equiv v_i(t) \quad (i = 1,2).$$

Thus the Stefan condition can be written in the form

$$(2.1.7)\qquad \dot{y}(t) = v_1(t) - v_2(t)$$

arising from $(2.1.4_6)$.

§2. The first boundary problem on a segment for a two-phase initial state and continuous agreement of boundary and initial conditions. Existence of the solution in the large

We now investigate problem B_1 on the segment $(l_1 = 0,\ l_2 = 1)$, supposing that conditions of continuous agreement of boundary and initial conditions prevail,

$$(2.2.1)\qquad f_1(t_0) = \phi_1(0);\quad \phi_1(l) = \phi_2(l) = 0;\quad f_2(t_0) = \phi_2(1).$$

Using the notation (0.2.5)—(0.2.7), we write

$$(2.2.2)\quad \begin{aligned} g_i(x,\xi,t) &= E(x-\xi,t) + (-1)^i E(x+\xi,t); \\ G_i(x,\xi,t) &= E(x-\xi,a^2t) + (-1)^i E(x+\xi-2,a^2t), \end{aligned}$$

where g_i and G_i are the Green's functions of the ith boundary problem for the equations $L_i u = 0$ $(i = 1, 2)$ on the halflines $x > 0$ and $x < 1$, respectively.

THEOREM 5. *The solution to problem* B_1 *under conditions* (2.2.1) *is equivalent to the solution of the system of integral equations*

$$
\begin{aligned}
v_1(t) &= 2\int_0^l \dot{\phi}_1(\xi) g_2(y(t), \xi, t - t_0)\, d\xi \\
&\quad - 2\int_{t_0}^t \dot{f}_1(\tau) g_2(y(t), 0, t - \tau)\, d\tau \\
&\quad - 2\int_{t_0}^t v_1(\tau) \frac{\partial}{\partial \xi} g_2(y(t), y(\tau), t - \tau)\, d\tau; \qquad (2.2.3_1)
\end{aligned}
$$

$$
\begin{aligned}
v_2(t) &= 2\int_l^1 \dot{\phi}_2(\xi) G_2(y(t), \xi, t - t_0)\, d\xi \\
&\quad + 2\int_{t_0}^t \dot{f}_2(\tau) G_2(y(t), 1, t - \tau)\, d\tau \\
&\quad + 2a^2 \int_{t_0}^t v_2(\tau) \frac{\partial}{\partial \xi} G_2(y(t), y(\tau), t - \tau)\, d\tau; \qquad (2.2.3_2)
\end{aligned}
$$

$$
y(t) = l + \int_{t_0}^t (v_1(\tau) - v_2(\tau))\, d\tau \qquad (2.2.3_3)
$$

in the sense that if v_1, v_2 *and* y *are bounded solutions of the system* $(2.2.3_1)$—$(2.2.3_3)$, *and* $u_i(x, t)$ $(i = 1, 2)$ *is determined from the equations*

$$
\begin{aligned}
u_1 &= \int_0^l \phi_1(\xi) g_1(x, \xi, t - t_0)\, d\xi + \int_{t_0}^t f_1(\tau) \frac{\partial}{\partial \xi} g_1(x, 0, t - \tau)\, d\tau \\
&\quad + \int_{t_0}^t v_1(\tau) g_1(x, y(\tau), t - \tau)\, d\tau; \qquad (2.2.4_1)
\end{aligned}
$$

$$
\begin{aligned}
u_2 &= \int_l^1 \phi_2(\xi) G_1(x, \xi, t - t_0)\, d\xi - a^2 \int_{t_0}^t f_2(\tau) \frac{\partial}{\partial \xi} G_1(x, 1, t - \tau)\, d\tau \\
&\quad - a^2 \int_{t_0}^t v_2(\tau) G_1(x, y(\tau), t - \tau)\, d\tau, \qquad (2.2.4_2)
\end{aligned}
$$

then u_1, u_2, y *is a solution of problem* B_1, *while, conversely, if* u_1, u_2, y *is a solution to problem* B_1, *and* v_1, v_2 *obey the equation* (2.1.7), *then* v_1, v_2 *and* y *constitute a solution to the system* $(2.2.3_1)$—$(2.2.3_3)$.

REMARK 1. If in addition to fulfilling conditions (2.2.1), u_1, u_2, y is a solution to problem B_1 and

$$q_i(x,t) = \partial u_i(x,t)/\partial x, \tag{2.2.5}$$

then the q_i are determined by the equations

$$\begin{aligned} q_1 = &- \int_{t_0}^{t} \dot{f}_1(\tau) g_2(x,0,t-\tau)\,d\tau + \int_0^l \dot{\phi}_1(\xi) g_2(x,\xi,t-t_0)\,d\xi \\ &- \int_{t_0}^{t} v_1(\tau) \frac{\partial}{\partial \xi} g_2(x,y(\tau),t-\tau)\,d\tau, \end{aligned} \tag{2.2.6$_1$}$$

$$\begin{aligned} q_2 = &\int_{t_0}^{t} \dot{f}_2(\tau) G_2(x,1,t-\tau)\,d\tau + \int_l^1 \dot{\phi}_2(\xi) G_2(x,\xi,t-t_0)\,d\xi \\ &+ a^2 \int_{t_0}^{t} v_2(\tau) \frac{\partial}{\partial \xi} G_2(x,y(\tau),t-\tau)\,d\tau; \end{aligned} \tag{2.2.6$_2$}$$

moreover q_1 and q_2 are continuous in D_1 and D_2, respectively.

REMARK 2. If condition (2.2.1) is not valid, but u_1, u_2 and y constitute a solution to problem B_1, then v_1 and v_2 are determined by equations (2.2.3$_i$) ($i = 1,2$) if to their right sides we add the term

$$2(\phi_1(0) - f_1(t_0)) g_2(y(t),0,t-t_0) - 2\phi_1(l) g_2(y(t),l,t-t_0) \tag{2.2.7$_1$}$$

and correspondingly

$$2(f_2(t_0) - \phi_2(1)) G_2(y(t),1,t-t_0) + 2\phi_2(l) G_2(y(t),l,t-t_0). \tag{2.2.7$_2$}$$

Simultaneously, we must add the terms

$$\begin{aligned} &(\phi_1(0) - f_1(t_0)) g_2(x,0,t-t_0) - \phi_1(l) g_2(x,l,t-t_0), \\ &(f_2(t_0) - \phi_2(1)) G_2(x,1,t-t_0) + \phi_2(l) G_2(x,l,t-t_0) \end{aligned} \tag{2.2.7$_3$}$$

to the right sides of equations (2.2.6$_1$)—(2.2.6$_2$).

REMARK 3. If u_1, u_2, u is a solution to problem A_1, and v_1, v_2 are defined in accordance with (2.2.7), then v_1, v_2, y is a solution to the system (2.2.3$_i$) ($i = 1,2,3$), if $l = 0$, $t_0 = 0$ and $\phi_2 = \phi$. Remark 2 remains valid with the corresponding changes.

THEOREM 6. *The solution to the system* (2.2.3$_1$)—(2.2.3$_3$) *satisfies the following conditions.*

a) *It exists and is unique in each interval* $t_0 < t < t_1$ *whose length tends to zero as*

$$\rho = \min\{\min(l, 1-l), K^{-1}\}, \tag{2.2.8}$$

where

$$K = \max\{\max_{t_0 \leqq t \leqq t_1; i=1,2} |\dot{f}_i(t)|; \max_{0<x<l} |\dot{\phi}_1(x)|; \max_{l<x<1} |\dot{\phi}_2(x)|\}. \tag{2.2.9}$$

b) $v_i(t)$ $(i = 1, 2)$ *satisfy the Lipschitz conditions*

$$|v_i(t) - v_i(\tau)| < \frac{A|t-\tau|}{\sqrt{t-t_0}+\sqrt{\tau-t_0}}; \quad t_0 < \tau \leqq t \leqq t_1, \tag{2.2.10}$$

where $A = \text{const}$ *depends only on* ρ *and on the length of the interval* (t_0, t_1).

c) *The solution of problem* B_1 *is stable in the sense of Theorem* 4, *Chapter* I.

The assertions of Theorems 5 and 6 are contained in those of Theorems 1—4 of Chapter I and therefore do not require proof.

REMARK. Using estimates derived in Chapter I in connection with the variations of the functional $I_{i,j}$, it is not difficult to prove that the solutions to equations (2.2.3$_i$) $(i = 1, 2)$ are differentiable for $t \geqq 0$.

Our next aim is to prove the existence of a solution to the problem B_1 "in the large" (i.e. on any arbitrary preassigned interval of time) and to investigate the asymptotic behavior of the solution for $t \to \infty$.

Following subsection B of §6, Chapter I, we shall define a stepwise process of constructing the solution to problem B_1 on the basis of Theorems 5 and 6; the step length is defined as the length Δt_0 of that interval of time (t_0, t_1) on which this theorem guarantees existence of the solution. Obviously, if this solution is constructed on the interval (t_0, t_1), then it can be defined on some interval (t_1, t_2). Similarly a construction of the solution can be realized for any finite number of steps.[1)]

Suppose that the solution is defined on an interval (t_0, T) and let ΔT be the length of the interval of definition of the solution in one step. If

---

[1)] The process of extending the solution is based on the following obvious assertion. Let $u(x, t)$ be a solution to the equation $Lu = 0$ in the parabolic domain $X_1(t) < x < X_2(t)$, $t > 0$, satisfying conditions of the first kind. Let u be defined for $0 \leqq t \leqq T$. Suppose that we consider the values $u(x, T)$ as initial data, and construct the solution to the same boundary problem in the domain $X_1(t) < x < X_2(t)$, $T < \tau \leqq t \leqq T_1$. Let us denote this solution by $u_1(x, t)$. We define

$$U(x, t) = \begin{cases} u(x, t) & 0 \leqq t \leqq T, \\ u_1(x, t) & T < t \leqq T_1. \end{cases}$$

Then U is a solution to the equation $LU = 0$ for $0 < t \leqq T_1$.

$$\inf_{t_0<T<\infty} \Delta T > 0 \tag{2.2.11}$$

then any time interval can be subdivided into a finite number of mesh steps, and consequently the solution to problem B_1 can be constructed on any arbitrary preassigned time interval.

From Theorem 6 it follows that ΔT tends to zero with

$$\rho(T) = \min\{\min(y(T), 1 - y(T)), N^{-1}(T)\}, \tag{2.2.12}$$

where

$$N(T) = \max\left\{\max_{t\in(T,T+\Delta T)} |f_i(t)|;\ \max_{0<x<y(T)} \left|\frac{\partial}{\partial x} u_1(x,T)\right|;\ \max_{y(T)<x<1} \left|\frac{\partial}{\partial x} u_2(x,T)\right|\right\}. \tag{2.2.13}$$

Hence in order to prove the existence of a solution "in the large" it suffices to obtain the a priori estimates

$$\inf\{y(t), 1 - y(t)\} > 0, \quad t_0 < t < \infty; \tag{2.2.14}$$

$$\sup_{t_0<t<\infty} \left\{\left|\frac{\partial u_1}{\partial x}\right|,\ 0 < x < y(t);\quad \left|\frac{\partial u_2}{\partial x}\right|,\ y(t) < x < 1\right\} < \infty. \tag{2.2.15}$$

The inequality (2.2.15) is obviously an implication of (2.2.14) and the a priori estimates

$$\sup \left|\frac{\partial}{\partial x} u_1(0,t)\right| < \infty; \quad \sup_{t_0<t<\infty} \left|\frac{\partial}{\partial x} u_2(1,t)\right| < \infty. \tag{2.2.16}$$

In fact, suppose that

$$\begin{aligned} &\inf|y(t), 1 - y(t)| = \alpha > 0; \quad t_0 \leqq t < \infty; \\ &\left|\frac{\partial}{\partial x} u_1(0,t)\right| < K; \quad \left|\frac{\partial}{\partial x} u_2(1,t)\right| < K; \quad t_0 \leqq t < \infty; \\ &\max_{0<x<l} \left|\frac{\partial}{\partial x} u_1(x,t_0)\right| < M; \quad \max_{l<x<1} \left|\frac{\partial}{\partial x} u_2(x,t_0)\right| < M. \end{aligned} \tag{2.2.17}$$

We assume that the solution to problem B_1 is defined on the interval (t_0, T). Let (t_k, t_{k+1}) $(k = 0, 1, \cdots)$ be the intervals in which $\dot{y}(t)$ has constant sign. For definiteness we assume that

$$\begin{aligned} &\dot{y}(t) \geqq 0 \text{ for } t_{2k} < t < t_{2k+1},\ k = 0, 1, 2, \cdots; \\ &\dot{y}(t) \leqq 0 \text{ for } t_{2k-1} < t < t_{2k},\ k = 1, 2, \cdots. \end{aligned} \tag{2.2.18}$$

We now define $\beta = \text{const} > 0$ by the conditions

$$\alpha\beta > \max_{t_0 < t < \infty;\, i=1,2} \max |f_i(t)|; \quad \beta \geqq \max\{M, K\} \tag{2.2.19}$$

and let

$$w = \beta(x - y(t)). \tag{2.2.20}$$

On the interval $t_0 < t < t_1$ we have

$$\begin{aligned} &w|_{x=0} = -\beta y(t) < -\beta\alpha < f_1(t); \\ &w|_{t=t_0} = \beta(x - l) < -M(l - x) < \phi_1(x) \text{ for } 0 < x < l = y(t_0); \\ &w|_{x=y(t)} = 0. \end{aligned} \tag{2.2.21}$$

Finally,

$$\frac{\partial^2 w}{\partial x^2} = 0; \quad \frac{\partial w}{\partial t} = -\beta\dot{y}(t) \leqq 0. \tag{2.2.22}$$

We now set

$$u_1^* = u_1 - w \quad (t_0 < t < t_1). \tag{2.2.23}$$

From the definition of u_1, and from (2.2.21) and (2.2.22), it follows that

$$L_1(u_1^*) \leqq 0; \quad u_1^*|_{x=0} \geqq 0; \quad u_1^*|_{t=t_0} \geqq 0; \quad u_1^*|_{x=y(t)} = 0. \tag{2.2.24}$$

Hence from the maximum principle we find that

$$u_1^* \geqq 0 \text{ for } 0 \leqq x \leqq y(t), \quad t_0 \leqq t \leqq t_1. \tag{2.2.25}$$

Since $u_1 = w = 0$ for $x = y(t)$, this implies that

$$\left.\frac{\partial u_1}{\partial x}\right|_{x=y(t)} \leqq \left.\frac{\partial w}{\partial x}\right|_{x=y(t)} = \beta. \tag{2.2.26}$$

At the same time, from (2.2.17) and (2.2.19) we conclude that

$$\left.\frac{\partial u_1}{\partial x}\right|_{x=0} \leqq \beta; \quad \left.\frac{\partial u_1}{\partial x}\right|_{t=t_0} \leqq \beta. \tag{2.2.27}$$

By Remark 1 to Theorem 5, $\partial u_1/\partial x$ is continuous for $0 \leqq x \leqq y(t)$, $t_0 \leqq t \leqq t_1$. Moreover $\partial u_1/\partial x$ is parabolic. Hence from (2.2.26) and (2.2.27) we conclude, by virtue of the maximum principle, that

$$\left|\frac{\partial u_1}{\partial x}\right| \leqq \beta \text{ for } 0 \leqq x \leqq y(t), \quad t_0 \leqq t \leqq t_1. \tag{2.2.28}$$

Furthermore from the Stefan condition (2.1.7) and from (2.2.28) it follows that[2)]

$$0 \leqq \frac{\partial u_2}{\partial x}\bigg|_{x=y(t)} \leqq \frac{\partial u_1}{\partial x}\bigg|_{x=y(t)} \leqq \beta, \quad t_0 \leqq t \leqq t_1. \tag{2.2.29}$$

If we examine w and u_2 on the interval $t_1 < t < t_2$ and reason analogously, we find that

$$\left|\frac{\partial u_2}{\partial x}\right| \leqq \beta \text{ for } y(t) \leqq x \leqq 1, \quad t_1 \leqq t \leqq t_2. \tag{2.2.30}$$

Hence from the Stefan condition and the nonpositivity of $\dot{y}(t)$ it follows that

$$\left|\frac{\partial u_1}{\partial x}\right| \leqq \beta \text{ for } 0 \leqq x \leqq y(t), \quad t_1 \leqq t \leqq t_2. \tag{2.2.31}$$

Repeating the argument, we find by induction that for arbitrary $k \geqq 0$,

$$\left|\frac{\partial u_1}{\partial x}\right|; \quad \left|\frac{\partial u_2}{\partial x}\right| \leqq \beta \text{ for } t_0 \leqq t \leqq t_k. \tag{2.2.32}$$

If the sequence $\{t_k\}$ has a point of accumulation $t = t^*$ then it is clear that (2.2.32) is valid for $t = t^*$. Thus the inequality (2.2.32) is realized for arbitrary t, as was asserted.[3)]

We now prove the validity of the estimate (2.2.16) under the assumption that (2.2.14) is valid. Suppose that u_1, u_2, y are defined for $t_0 \leqq t \leqq T$ and that α is defined in agreement with (2.2.17). We choose functions $\zeta_i(t)$ $(i = 1, 2)$, satisfying the conditions

c—1: $\zeta_i(t)$ is continuous together with its first derivative for $t_0 \leqq t \leqq T$.

c—2:

$$\zeta_i(t_0) = 0; \quad 0 < \zeta_i(t) < \alpha, \quad t_0 < t \leqq T, \quad (i = 1, 2); \tag{2.2.33}$$

c—3: For arbitrary $t \in (t_0, T)$ we have

$$\begin{aligned} &1 - \zeta_1(t)\,|\,\dot{\zeta}_1(t)\,| - \zeta_1^2(t)\,|f_1(t)\,|^{-1}|\dot{f}_1(t)\,| \geqq 0; \\ &1 - a^{-2}\zeta_2(t)\,|\,\dot{\zeta}_2(t)\,| - a^2\zeta_2^2(t) f_2^{-1}(t)\,|\dot{f}_2(t)\,| \geqq 0. \end{aligned} \tag{2.2.33*}$$

[2)] From the conditions $(2.1.1_2)$—$(2.1.1_3)$ and the maximum principle it follows that $u_2 \geqq 0$. Consequently $\partial u_2/\partial x|_{x=y(t)} \equiv v_2(t) \geqq 0$.

[3)] This shows that t^* is attainable by a finite number of steps. By our proof $\dot{y}(t)$ is differentiable on each step. Consequently t^* cannot be a point of accumulation of zeros of $\dot{y}(t)$.

Moreover, assume that

$$
\begin{aligned}
u_1^*(x,t) &= (e-1)^{-1} f_1(t)\,[\exp(1 - x\zeta_1^{-1}(t)) - 1];\\
u_2^*(x,t) &= (e-1)^{-1} f_2(t)\,[\exp(1 - (1-x)\,\zeta_2^{-1}(t)) - 1].
\end{aligned}
\tag{2.2.34}
$$

From (2.1.1) and (2.2.33*) it follows that u_1^* is superparabolic and u_2^* is subparabolic for $0 < x < \zeta(t)$, $t_0 < t < T$ and $1 - \zeta_2(t) < x < 1$, $t_0 < t < T$, respectively. Further, from (2.2.34) we find that

$$
\begin{aligned}
u_1^*|_{x=\zeta_1(t)} &= 0; \quad u_1^*|_{x=0} = f_1(t) < 0;\\
u_2^*|_{x=1-\zeta_2(t)} &= 0; \quad u_2^*|_{x=1} = f_2(t) > 0.
\end{aligned}
\tag{2.2.35}
$$

Finally, from the boundary conditions of problem B_1,

$$
\begin{aligned}
&u_1(0,t) = f_1(t) < 0; \quad u_1(x,t_0) = \phi_1(x) \leqq 0; \quad u_1(y(t),t) = 0;\\
&u_2(1,t) = f_2(t) > 0; \quad u_2(x,t_0) = \phi_2(x) \geqq 0; \quad u_2(y(t),t) = 0,
\end{aligned}
\tag{2.2.36}
$$

the inequalities

$$
y(t) \geqq \zeta_1(t); \quad y(t) \leqq 1 - \zeta_2(t); \quad t \geqq t_0
\tag{2.2.37}
$$

and the maximum principle, it follows that

$$
u_1|_{x=\zeta_1(t)} \leqq 0; \quad u_2|_{x=1-\zeta_2(t)} \geqq 0; \quad t > t_0.
\tag{2.2.38}
$$

We now set

$$
\tilde{u}_1 = u_1^* - u_1; \quad \tilde{u}_2 = u_2^* - u_2.
\tag{2.2.39}
$$

From the parabolicity of u_i $(i = 1,2)$, the superparabolicity of u_1^* and the subparabolicity of u_2^* we infer the superparabolicity of u_1 and the subparabolicity of u_2. Referring to (2.2.35), (2.2.36) and (2.2.38), we therefore find that

$$
\left.
\begin{aligned}
&u_1(x,t) \leqq u_1^*(x,t) \text{ for } 0 \leqq x < \zeta_1(t)\\
&u_2^*(x,t) \leqq u_2(x,t) \text{ for } 1 - \zeta_2(t) < x < 1
\end{aligned}
\right\} \; t_0 \leqq t \leqq T.
\tag{2.2.40}
$$

Hence from (2.2.35) and (2.2.36) it follows that

$$
\begin{aligned}
\frac{\partial}{\partial x} u_1(0,t) &\leqq \frac{\partial}{\partial x} u_1^*(0,t) = -f_1(t)\,(e-1)^{-1} e \cdot \zeta_1^{-1}(t);\\
\frac{\partial}{\partial x} u_2(1,t) &\leqq \frac{\partial}{\partial x} u_2^*(1,t) = f_2(t)\,(e-1)^{-1} e \cdot \zeta_2^{-1}(t).
\end{aligned}
\tag{2.2.41}
$$

Now let t_0^* be some value of $t > t_0$ for which the solution u_1, u_2, y of problem B_1 is defined by virtue of Theorem 6. Using the fact that

(2.2.42) $$0 < \delta \leqq \zeta_i(t) \text{ for } t_0^* < t < T \quad (i = 1, 2),$$

we find

(2.2.43) $$\sup_{t_0^* < t \leqq T} \left| \frac{f_i(t)}{\zeta_i(t)} \right| \leqq K^* \quad (i = 1, 2),$$

where $K^* > 0$ is a suitably chosen constant. From (2.2.41) it follows that for every $t \in (t_0^*, T)$,

(2.2.44) $$\frac{\partial}{\partial x} u_1(0, t) \leqq K^*; \quad \frac{\partial}{\partial x} u_2(1, t) \leqq K^*.$$

Since $\partial u_1(0,t)/\partial x$ and $\partial u_2(1,t)/\partial x$ are automatically bounded for $t_0 \leqq t \leqq t_0^*$, we see that K^* can be chosen so small that (2.2.44) is satisfied for every $t \in (t_0, T)$.

We now estimate $\partial u_1(0,t)/\partial x$ and $\partial u_2(0,t)/\partial x$ from below. By the maximum principle and the conditions of problem B_1 it follows that

(2.2.45) $$\frac{\partial}{\partial x} u_i(y(t), t) = v_i(t) \geqq 0 \quad (i = 1, 2)$$

for every $t \geqq t_0$. By the definition of g_i and G_i it follows that

(2.2.46) $$\begin{aligned} &-\frac{\partial}{\partial \xi} g_2(0, y(\tau), t - \tau) \geqq 0 \text{ for } y(\tau) \geqq 0; \\ &\frac{\partial}{\partial \xi} G_2(1, y(\tau), t - \tau) \geqq 0 \text{ for } 1 - y(\tau) \geqq 0. \end{aligned}$$

Comparing (2.2.45), (2.2.46), ($2.2.6_1$) and (2.2.5), we obtain the estimate

(2.2.47) $$\frac{\partial}{\partial x} u_1(0, t) \geqq -\frac{1}{\sqrt{\pi}} \int_0^t \frac{\dot{f}_1(\tau)\, d\tau}{\sqrt{t - \tau}} + \int_0^l \dot{\phi}_1(\xi) g_2(0, \xi, t)\, d\xi.$$

An analogous estimate holds for $\partial u_2(1,t)/\partial x$.

Thus $\partial u_1(0,t)/\partial x$ and $\partial u_2(1,t)/\partial x$ are bounded from below, independently of $y(t)$ and $v_i(t)$ $(i = 1, 2)$. But, together with (2.2.44), this implies that there exists some $K > 0$ such that

(2.2.48) $$\left| \frac{\partial}{\partial x} u_1(0, t) \right|, \quad \left| \frac{\partial}{\partial x} u_2(1, t) \right| \leqq K \text{ for } t \in (t_0, T),$$

where K can be chosen independently of T if α in (2.2.33) is independent of T. Thus in order to prove the existence of a solution to problem B_1

"in the large" it remains for us merely to prove the estimate

$$\inf_{t_0<t<\infty} (y(t), 1-y(t)) = \alpha > 0. \tag{2.2.49}$$

However, this is an obvious consequence of the following elementary lemma.

LEMMA 1. *Let* u_1, u_2, y *be a solution to problem* B_1. *Moreover, suppose that* $z_j(t)$ $(j=1,2)$ *are differentiable for* $t_0 \leqq t \leqq T$ *and* $l_1 = z_1(t_0) \leqq l \leqq z_2(t_0) = l_2$. *Define* $u_{i,1}$ *by the following conditions:* $u_{1,1}$ *is superparabolic in* $D_{1,1}(z_1, t_0, 0, l_1)$ *with*

$$\begin{aligned} &f_1(t) \leqq u_{1,1}(0,t) \leqq 0 \text{ for } t_0 \leqq t \leqq T; \\ &\phi_1(x) \leqq u_{1,1}(x,t_0) \leqq 0 \text{ for } 0 \leqq x \leqq l_1; \end{aligned} \tag{2.2.50}$$

$u_{2,1}$ *is superparabolic in* $D_{2,1}(z_1, t_0, 1, l_1)$ *with*

$$\begin{aligned} &0 \leqq f_2(t) \leqq u_2(1,t) \text{ for } t_0 \leqq t \leqq T; \\ &0 \leqq \phi_2(x) \leqq u_2(x,t_0) \text{ for } l \leqq x \leqq 1. \end{aligned} \tag{2.2.51}$$

Then, if for every $t \in (t_0, T)$ *the inequality*

$$\dot{z}_1(t) < \left(\frac{\partial}{\partial x} u_{1,1} - \frac{\partial}{\partial x} u_{2,1}\right)\Bigg|_{x=z_1(t)} \tag{2.2.52}$$

is valid, and if there exists some t_0^* *such that on the interval* (t_0, t_0^*)

$$z_1(t) \leqq y(t), \tag{2.2.53}$$

then (2.2.53) *is valid for every* $t \in (t_0, T)$.

Analogously, if $u_{1,2}$ *is subparabolic in* $D_1(z_2, t_0, 0, l_2)$ *and* $u_{2,2}$ *is superparabolic in* $D_2(z_2, t_0, 1, l_2)$ *with*

$$\begin{aligned} &u_{1,2}(0,t) \leqq f_1(t) \leqq 0; \\ &0 \leqq u_{2,2}(1,t) \leqq f_2(t) \text{ for } t_0 \leqq t \leqq T; \\ &u_{1,2}(x,t_0) \leqq \phi_1(x) \leqq 0 \text{ for } 0 \leqq x \leqq l; \\ &0 \leqq u_{2,2}(x,t_0) \leqq \phi_2(x) \text{ for } l_2 \leqq x \leqq 1, \end{aligned} \tag{2.2.54}$$

then the inequalities

$$\begin{aligned} &\dot{z}_2(t) > \left(\frac{\partial}{\partial x} u_{1,2} - \frac{\partial}{\partial x} u_{2,2}\right)\Bigg|_{x=z_2(t)} \text{ for } t_0 \leqq t \leqq T; \\ &y(t) \leqq z_2(t) \text{ for } t_0 \leqq t \leqq t_0^* \end{aligned} \tag{2.2.55}$$

imply

$$y(t) \leqq z_2(t) \quad \textit{for} \quad t_0 \leqq t \leqq T. \tag{2.2.56}$$

REMARK 1. The lemma is valid for $y(t_0) = 0$ $(y(t_0) = 1)$. In this case $z_1(t_0) = y(t_0)$ $(z_2(t_0) = y(t_0))$. Consequently, in this case, in order to make effective use of Lemma 1 it is necessary to investigate independently the character of the growth of $y(t)$ at the time $t = t_0$.

REMARK 2. The lemma remains valid if we replace the conditions of problem B_1 by those of problem C. It is only necessary to require that u_1 and u_2 converge uniformly for $|x| \to \infty$ to a limiting value $\phi_i(x)$ and that the $u_{i,j}$ have a definite limit at infinity.

Let us assume that Lemma 1 is proved. Let

$$\inf_{t_0 < t < \infty} |f_i(t)| = m_i > 0; \quad \sup_{t_0 < t < \infty} |f_i(t)| = M_i \quad (i = 1, 2). \tag{2.2.57}$$

We set

$$u_{1,1} = -m_1^* \left(1 - \frac{x}{l_1}\right); \quad u_{2,1} = M_2^* \left(1 - \frac{1-x}{1-l_1}\right) \tag{2.2.58}$$

and require that

$$\begin{gathered} -m_1^* \left(1 - \frac{x}{l_1}\right) \geqq \phi_1(x) \text{ for } 0 < x < l_1 < y(0); \\ M_2^* \left(1 - \frac{1-x}{1-l_1}\right) \geqq \phi_2(x) \text{ for } y(0) = l < x < 1; \\ \frac{m_1^*}{l_1} \geqq \frac{M_2^*}{1-l_1}; \quad m_1^* < m_1; \quad M_2^* > M_2. \end{gathered} \tag{2.2.59}$$

The compatibility conditions (2.2.59) are obvious by virtue of the condition

$$\min|\phi_1(x)| > 0 \text{ for } 0 \leqq x \leqq l_1 < l, \tag{2.2.60}$$

stemming from (2.1.2), and the boundedness of $\phi_2(x)$. From the definitions (2.2.58) and from (2.2.59) it follows clearly that for all $t \geqq t_0$ the functions $u_{1,1}$, $u_{2,1}$ and $z_1 \equiv l_1$ satisfy every condition of Lemma 1, and hence for every $t \geqq t_0$

$$y(t) \geqq l_1. \tag{2.2.61}$$

Similarly we can prove the existence of an $l_2 > 0$ such that

$$y(t) \leqq l_2 \tag{2.2.62}$$

for every $t \geqq t_0$.

Thus it remains for us to prove Lemma 1.

By assumption the inequality $z_1(t) \leqq y(t)$ is valid on some interval (t_0, t_0^*). We assume that the set $\mathscr{E}$ of all $t \in (t_0, T)$ for which $y(t) < z_1(t)$ is not empty. Let us set

$$\tau_0 = \inf \mathscr{E}. \tag{2.2.63}$$

Then $\tau_0 \geqq t_0^* > t_0$. From the definition of τ_0 it follows that

$$y(\tau_0) = z_1(\tau_0); \quad y(t) > z_1(t) \text{ for } t_0 < t < \tau_0 \tag{2.2.64}$$

and for any $\delta > 0$ there is a point τ on the interval $(\tau_0, \tau_0 + \delta)$ such that

$$y(\tau) < z_1(\tau). \tag{2.2.65}$$

Thus on the interval (t_0, τ_0) the function u_1 is defined in the domain of existence of $u_{1,1}$, and $u_{2,1}$ is defined in the domain of existence of u_2. Let

$$\begin{aligned} w_{1,1} &= u_{1,1} - u_1; \\ w_{2,1} &= u_{2,1} - u_2. \end{aligned} \tag{2.2.66}$$

From the definitions of $u_{i,1}$ and u_i $(i = 1, 2)$ we see that $w_{i,1}$ is superparabolic everywhere in its domain of definition. Moreover, from the boundary conditions (2.2.50) and (2.2.51), as well as from the maximum principle, it follows that

$$\begin{aligned} &w_{1,1}(0,t) \geqq 0; \quad w_{1,1}(x,t_0) \geqq 0; \\ &\quad w_{1,1}(z_1(t),t) = -u_1(z_1(t),t) \geqq 0; \\ &w_{2,1}(1,t) \geqq 0; \quad w_{2,1}(x,t_0) \geqq 0; \\ &\quad w_{2,1}(y(t),t) = u_{2,1}(y(t),t) \geqq 0, \end{aligned} \tag{2.2.67}$$

whence from the superparabolicity of $w_{i,1}$ it follows that throughout the domain of definition

$$w_{i,1}(x,t) \geqq 0 \quad (i = 1, 2). \tag{2.2.68}$$

Since in addition

$$w_{1,1}(z_1(\tau_0), \tau_0) = 0; \quad w_{2,1}(y(\tau_0), \tau_0) = 0; \quad y(\tau_0) = z_1(\tau_0), \tag{2.2.69}$$

it is then obvious that

$$\begin{aligned} &\frac{\partial}{\partial x} u_1(y(\tau_0), \tau_0) \geqq \frac{\partial}{\partial x} u_{1,1}(z_1(\tau_0), \tau_0); \\ &\frac{\partial}{\partial x} u_2(y(\tau_0), \tau_0) \leqq \frac{\partial}{\partial x} u_{2,1}(z_1(\tau_0), \tau_0). \end{aligned} \tag{2.2.70}$$

Comparing (2.2.70), (2.2.52) and the Stefan condition (2.1.7), we find that

$$\begin{aligned} \dot{y}(\tau_0) &= \frac{\partial}{\partial x} u_1(y(\tau_0), \tau_0) - \frac{\partial}{\partial x} u_2(y(\tau_0), \tau_0) \\ &\geqq \frac{\partial}{\partial x} u_{1,1}(z_1(\tau_0), \tau_0) - \frac{\partial}{\partial x} u_{2,1}(z_1(\tau_0), \tau_0) > \dot{z}_1(\tau_0). \end{aligned} \tag{2.2.71}$$

By virtue of the fact that $y(\tau_0) = z_1(\tau_0)$ this proves the existence of some $\delta > 0$ such that

$$y(t) \geqq z_1(t) \quad \text{for } t \in (\tau_0, \tau_0 + \delta), \tag{2.2.72}$$

contradicting (2.2.65). But this shows that $\mathscr{S}$ is really a null set, and this proves the validity of inequality (2.2.53). The inequality (2.2.56) is proved analogously. Thus the lemma is proved.

§3. Asymptotic behavior of the solution to problem B_1

We now prove that, under the assumption (2.1.1) for the behavior of the functions f_1 and f_2, the solution of problem B_1 tends asymptotically to the solution of the corresponding stationary problem, i.e. that the following theorem is valid.

THEOREM 7. *If*

$$\lim_{t \to \infty} f_1(t) = -T_1 < 0; \quad \lim_{t \to \infty} f_2(t) = T_2 > 0, \tag{2.3.1}$$

then as $t \to \infty$ *the solutions* $u_1(x,t)$, $u_2(x,t)$, $y(t)$ *of problem* B_1 *tend (uniformly in* x*) to the limits*

$$\lim_{t \to \infty} u_i(x,t) = U(x) = -T_1 + (T_2 + T_1)x \quad (i = 1,2); \tag{2.3.2}$$

$$\lim_{t \to \infty} y(t) = L = \frac{T_1}{T_1 + T_2}. \tag{2.3.3}$$

Using Lemma 1 we note first that to prove (2.3.3) it suffices to prove the following lemma.

LEMMA 2. *For any $\epsilon > 0$ there exists some $t_0^* \geqq t_0$ and $t_1 \geqq t_0^*$ such that any $t_2 \geqq t_1$ can be set into the corresponding functions $u_{i,j}(x,t)$ and $z_j(t)$ $(i,j = 1,2)$, and these will satisfy the conditions A_j:*

$$A_1\begin{cases}
(1)\quad z_1(t_0^*) = 0; \quad z_1(t_2) = L - \epsilon; \\
(2)\quad L_1 u_{1,1} \leqq 0 \ \textit{for}\ 0 < x < z_1(t); \quad t_0^* < t \leqq t_2; \\
(3)\quad L_2 u_{2,1} \leqq 0 \ \textit{for}\ z_1(t) < x < 1; \quad t_0^* < t \leqq t_2; \\
(4)\quad \begin{cases} u_{1,1}(0,t) \geqq f_1(t); \quad u_{i,1}(z_1(t),t) = 0 \quad (i = 1,2); \\ \qquad\qquad\qquad\qquad u_{2,1}(1,t) \geqq f_2(t); \quad t_0^* < t \leqq t_2; \end{cases} \\
(5)\quad u_{2,1}(x,t_0^*) \geqq u_2(x,t_0^*) \ \textit{for}\ y(t_0^*) < x < 1; \\
(6)\quad \dot{z}_1(t) < \left(\dfrac{\partial}{\partial x} u_{1,1} - \dfrac{\partial}{\partial x} u_{2,1}\right)\Big|_{x=z_1(t)}; \quad t_0^* < t \leqq t_2;
\end{cases}$$

$$A_2\begin{cases}
(1)\quad z_2(t_0^*) = 1; \quad z_2(t_2) = L + \epsilon; \\
(2)\quad L_1 u_{1,2} \geqq 0 \ \textit{for}\ 0 < x < z_2(t); \quad t_0^* < t \leqq t_2; \\
(3)\quad L_2 u_{2,2} \geqq 0 \ \textit{for}\ z_2(t) < x < 1; \quad t_0^* < t \leqq t_2; \\
(4)\quad \begin{cases} u_{1,2}(0,t) \leqq f_1(t); \quad u_{i,2}(z_2(t),t) = 0 \quad (i = 1,2); \\ \qquad\qquad\qquad\qquad u_{2,2}(1,t) \leqq f_2(t); \quad t_0^* < t \leqq t_2; \end{cases} \\
(5)\quad u_{1,2}(x,t_0^*) \leqq u_1(x,t_0^*) \ \textit{for}\ 0 < x < y(t_0^*); \\
(6)\quad \dot{z}_2(t) > \left(\dfrac{\partial}{\partial x} u_{1,2} - \dfrac{\partial}{\partial x} u_{2,2}\right)\Big|_{x=z_2(t)}; \quad t_0^* < t \leqq t_2.
\end{cases}$$

In fact, by the definition of $u_{i,j}$ and z_j, these functions satisfy every condition of Lemma 1, and consequently

$$z_1(t) \leqq y(t) \leqq z_2(t) \ \text{ for } t \in (t_0^*, t_2). \tag{2.3.4}$$

In particular,

$$z_1(t_2) = L - \epsilon \leqq y(t_2) \leqq L + \epsilon = z_2(t_2). \tag{2.3.4*}$$

Since $t_2 > t_1$ is arbitrary, this shows that for any $\epsilon > 0$ there exists some $t_1 > 0$ so large that

$$L_1 - \epsilon \leqq y(t) \leqq L_1 + \epsilon \ \text{ for every } t > t_1, \tag{2.3.4**}$$

which proves our assertion.

Thus, let $\epsilon > 0$ be arbitrarily small. Choose η such that

$$\eta < (T_1 + T_2)\,\epsilon/2, \tag{2.3.5}$$

and set

$$T_{1,1} = T_1 - \eta; \quad T_{2,1} = T_2 + \eta, \tag{2.3.6}$$

where T_1 and T_2 are defined in agreement with (2.3.1).

Now choose t_0^* so large that for every $t \geqq t_0^*$ we have

$$f_1(t) = u_1(0,t) \leqq -T_{1,1}; \quad f_2(t) = u_2(1,t) \leqq T_{2,1} \tag{2.3.7$_1$}$$

and in addition

$$u_2(x,t_0^*) \leqq T_{2,1} \text{ for } y(t_0^*) \leqq x \leqq 1. \tag{2.3.7$_2$}$$

The possibility of choosing such a t_0^* is guaranteed by condition (2.3.1).[4)]

For the sake of simplicity we set $t_0^* = 0$ in the sequel.

Let γ_1^* satisfy the inequality

$$\gamma_1^* < \min\left\{\sqrt{\frac{(T_1+T_2)\epsilon}{4(T_1+3(T_2+\eta)/a^2(L-\epsilon))}};\right. \\ \left.\sqrt{\frac{(T_1+T_2)\epsilon}{8(1-L+\epsilon)}}; \frac{a(L-\epsilon)\ln^{1/2}3}{\sqrt{1-(L-\epsilon)^2}}\right\}, \tag{2.3.8}$$

and set

$$t_1 = \frac{(L-\epsilon)^2}{4\gamma_1^{*2}}. \tag{2.3.9}$$

We now choose an arbitrary $t_2 > t_1$. Set

$$\gamma_1 = \frac{L-\epsilon}{2\sqrt{t_2}}; \quad z_1 = 2\gamma_1\sqrt{t}; \\ u_{1,1}(x,t) = -T_{1,1}\left(1 - \frac{\operatorname{erf}(x/2\sqrt{t})}{\operatorname{erf}\gamma_1}\right); \tag{2.3.10} \\ u_{2,1}(x,t) = T_{2,1}\frac{\operatorname{erf}(x/2a\sqrt{t}) - \operatorname{erf}(\gamma_1/a)}{\operatorname{erf}(1/2a\sqrt{t}) - \operatorname{erf}(\gamma_1/a)}.$$

It is easily seen that $u_{i,1}$, z_1 $(i = 1,2)$ satisfy every condition A_1 of Lemma 2. In particular,

$$u_{1,1}(0,t) = -T_{1,1} \geqq u_1(0,t); \quad u_{1,1}(z_1(t),t) = 0; \\ u_{2,1}(1,t) = T_{2,1} \geqq u_2(1,t); \quad u_{2,1}(x,0) = T_{2,1} \geqq u_2(x,0). \tag{2.3.11}$$

[4)] See the proof of (2.3.2) below, and also Tihonov [177].

Further,

$$L_1 u_{1,1} \equiv 0;$$

$$L_2 u_{2,1} \equiv -T_{2,1} \frac{\operatorname{erf}(x/2a\sqrt{t}) - \operatorname{erf}(\gamma_1/a)}{(\operatorname{erf}(\gamma_1/a) - \operatorname{erf}(1/2a\sqrt{t}))^2} \cdot \frac{\exp(-1/4a^2t)}{2a\sqrt{\pi}t^{3/2}} \leqq 0 \tag{2.3.11*}$$

for $x \geqq z_1(t)$.

Thus it is only necessary to prove that for every $t \in (0, t_2)$

$$\dot{z}_1(t) = \frac{\gamma_1}{\sqrt{t}} < \frac{T_{1,1}\exp(-\gamma_1^2)}{\sqrt{\pi t}\operatorname{erf}\gamma_1} - \frac{T_{2,1}\exp(-\gamma_1^2/a^2)}{a\sqrt{\pi t}(\operatorname{erf}(1/2a\sqrt{t}) - \operatorname{erf}(\gamma_1/a))}$$
$$= \left(\frac{\partial}{\partial x} u_{1,1} - \frac{\partial}{\partial x} u_{2,1}\right)\Big|_{x=z_1(t)}; \quad 0 < t \leqq t_2. \tag{2.3.12}$$

Since when $0 \leqq t \leqq t_2$ we have

$$\operatorname{erf}\frac{1}{2a\sqrt{t}} \geqq \operatorname{erf}\frac{1}{2a\sqrt{t_2}} = \operatorname{erf}\frac{\gamma_1}{a(L-\epsilon)}, \tag{2.3.13}$$

it is sufficient to show that

$$\gamma_1 < \frac{T_{1,1}\exp(-\gamma_1^2)}{\sqrt{\pi}\operatorname{erf}\gamma_1} - \frac{T_{2,1}\exp\left(-\frac{\gamma_1^2}{a^2}\right)}{a\sqrt{\pi}\left(\operatorname{erf}\frac{\gamma_1}{a(L-\epsilon)} - \operatorname{erf}\frac{\gamma_1}{a}\right)} \equiv f(\gamma_1). \tag{2.3.14}$$

Using the theorem of the mean, we find that

$$f(\gamma_1) = \frac{T_{1,1}(1+\epsilon-L)\exp(-\gamma_1^2(1-\Theta_1^2))}{2\gamma_1(1-L+\epsilon)}$$
$$- \frac{T_{2,1}(L-\epsilon)\exp[-(\gamma_1^2/a^2)(1-\Theta_2^2/(L-\epsilon)^2)]}{2\gamma_1(1-L+\epsilon)}. \tag{2.3.15}$$

Here

$$0 < \Theta_1 < 1; \quad L - \epsilon < \Theta_2 < 1. \tag{2.3.16}$$

But

$$\exp(-\gamma_1^2(1-\Theta_1^2)) > 1 - \gamma_1^2; \tag{2.3.17$_1$}$$

$$\exp\left[-\frac{\gamma_1^2}{a^2}\left(1 - \frac{\Theta_2^2}{(L-\epsilon)^2}\right)\right] < 1 + \frac{3\gamma_1^2}{a^2(L-\epsilon)^2}. \tag{2.3.17$_2$}$$

The last inequality holds by virtue of (2.3.16) and (2.3.8), since from (2.3.9) and (2.3.10) we can conclude that

$$\gamma_1 \leqq \gamma_1^* \text{ for } t_2 > t_1. \tag{2.3.17$_3$}$$

Thus

$$f(\gamma_1) > \frac{T_{1,1}(1+\epsilon-L) - T_{2,1}(L-\epsilon) - \gamma_1^2(T_{1,1} + 3T_{2,1}/a^2(L-\epsilon))}{2\gamma_1(1-L+\epsilon)}. \tag{2.3.18}$$

Using (2.3.3), (2.3.5), (2.3.6), (2.3.17$_3$) and (2.3.8), we find now that

$$\begin{aligned} f(\gamma_1) &> \frac{(T_1+T_2)\epsilon - \eta - \gamma_1^2[T_1 + 3(T_2+\eta)/a^2(L-\epsilon)]}{2\gamma_1(1+\epsilon-L)} \\ &> \frac{(T_1+T_2)\epsilon}{8\gamma_1(1+\epsilon-L)}. \end{aligned} \tag{2.3.19}$$

However, by virtue of (2.3.17$_3$) and (2.3.8) this shows that

$$\gamma_1 < f(\gamma_1), \tag{2.3.20}$$

as was asserted.

The verification of the existence of $u_{1,2}$, $u_{2,2}$ and z_2 satisfying the conditions A_2 of Lemma 2 is completely analogous.

Thus we have proved that $\lim_{t\to\infty} y(t)$ exists, and moreover,

$$\lim_{t\to\infty} y(t) = L = \frac{T_1}{T_1+T_2}. \tag{2.3.21}$$

In addition, it is possible to verify that

$$\lim_{t\to\infty} \dot{y}(t) = 0. \tag{2.3.22}$$

In fact, by virtue of Theorem 6, $v_i(t) = \partial u_i(y(t),t)/\partial x$ satisfies a Hölder condition of the form

$$|v_i(t) - v_i(\tau)| < A\sqrt{t-\tau} \quad (i = 1,2) \tag{2.3.23}$$

with the constant A dependent only on the term ρ from the conditions (2.2.8), and on the length Δt of the interval on which the solution to the problem was constructed by Theorem 6. By virtue of results of §2 (see (2.2.14) and (2.2.15)) the term ρ is defined independently of the length of the interval on which the solution is defined by finitely many steps; hence it is also independent of the length of the interval of definition of the solution. Thus we see that A in (2.3.23), as defined,

is independent of the quantity t. But this is clearly sufficient to conclude that (2.3.22) is valid, since $y(t)$ has a limit for $t \to \infty$.

We now prove assertion (2.3.2) of Theorem 7. Let $\epsilon > 0$ be arbitrarily small. We choose t_0^* so large that for every $t \geqq t_0^*$

$$|f_1(t) + T_1| < \frac{\epsilon}{2}; \quad \left| T_1\left(1 - \frac{y(t)}{L}\right)\right| < \frac{\epsilon}{2}. \tag{2.3.24}$$

This is possible by virtue of (2.3.1) and (2.3.3).

We now set

$$u_1 = -T_1\left(1 - \frac{x}{L}\right) + w_1(x,t) \tag{2.3.25}$$

and consider t_0^* as the initial moment in time. From the definition of u_1 it follows that

$$\begin{aligned} &L_1 w_1 = 0; \quad w_1(0,t) = f_1(t) + T_1; \\ &w_1(y(t),t) = T_1\left(1 - \frac{y(t)}{L}\right); \\ &w_1(x,t_0^*) = u_1(x,t_0^*) + T_1\left(1 - \frac{x}{L}\right) \text{ for } 0 < x < y(t_0^*). \end{aligned} \tag{2.3.26}$$

We now write w_1 as the sum of two functions w_1' and w_1'' satisfying the conditions

$$\begin{aligned} &L_1 w_1' = 0; \quad w_1'(x,t_0^*) = w_1(x,t_0^*); \\ &w_1'(0,t) = 0; \quad w_1'(y(t),t) = 0; \quad t > t_0^*; \end{aligned} \tag{2.3.27$_1$}$$

$$\begin{aligned} &L_1 w_1'' = 0; \quad w_1''(x,t_0^*) = 0; \\ &w_1''(0,t) = w_1(0,t); \quad w_1''(y(t),t) = w_1(y(t),t). \end{aligned} \tag{2.3.27$_2$}$$

From a comparison of (2.3.27$_i$) ($i = 1, 2$), (2.3.26) and (2.3.24) we see that

$$|w_1''(y(t),t)| < \frac{\epsilon}{2}; \quad |w_1''(0,t)| < \frac{\epsilon}{2}. \tag{2.3.28}$$

Hence from (2.3.27$_1$), by virtue of the maximum principle, it follows that throughout its domain of definition

$$|w_1''| < \epsilon/2. \tag{2.3.29}$$

On the other hand we see from its definition that the absolute value of w_1' is majorized by the integral

$$\int_0^{y(t_0^*)} |w_1(\xi, t_0^*)|\, E(x - \xi, t - t_0^*)\, d\xi. \tag{2.3.30}$$

But this shows that there is some time $t_1 > t_0^*$ such that everywhere in its domain of definition

$$|w_1'| < \epsilon/2 \text{ for } t > t_1. \tag{2.3.31}$$

By comparing the assertions (2.3.31), (2.3.29), the equation

$$w_1 = w_1' + w_1'' \tag{2.3.32}$$

and (2.3.25), we see that the assertion

$$\lim_{t \to \infty} u_1(x,t) = -T_1\left(1 - \frac{x}{L}\right). \tag{2.3.33}$$

is valid. The validity of the same equation for u_2 is established in the same way. The theorem is thus proved.

§ 4. Degeneration of one of the phases for $t > t_0$

Up to now problem B_1 has been investigated under the assumption that two phases exist for every $t > t_0$, thus guaranteeing the condition

$$\min|f_i(t)| > 0, \quad t_0 \leqq t < \infty; \quad i = 1,2.$$

We now examine the following form of the Stefan problem:

$$\begin{aligned} &L_1 u_1 = 0 \text{ for } 0 < x < y_1(t); \\ &L_1 u_3 = 0 \text{ for } y_2(t) < x < 1; \quad t > 0; \\ &L_2 u_2 = 0 \text{ for } y_1(t) < x < y_2(t); \end{aligned} \tag{2.4.1$_1$}$$

$$u_1(0,t) = f_1(t) \leqq 0; \quad u_3(1,t) = f_3(t) \geqq 0; \quad t > 0; \tag{2.4.1$_2$}$$

$$u_i(y_j(t),t) = 0; \quad i = 1,2 \text{ for } j = 1; \quad i = 2,3 \text{ for } j = 2;$$

$$u_i(x,0) = \phi_i(x); \quad \phi_i(x) \leqq 0 \text{ for } i = 1,3; \quad \phi_2(x) \geqq 0; \tag{2.4.1$_3$}$$

$$\begin{aligned} \dot{y}_1(t) &= \left(\frac{\partial}{\partial x} u_1 - \frac{\partial}{\partial x} u_2\right)\Big|_{x=y_1(t)}; \quad y_1(0) = l_1; \\ \dot{y}_2(t) &= \left(\frac{\partial}{\partial x} u_3 - \frac{\partial}{\partial x} u_2\right)\Big|_{x=y_2(t)}; \quad y_2(0) = l_2, \end{aligned} \tag{2.4.1$_4$}$$

where

$$0 < l_1 < l_2 < 1, \tag{2.4.1$_5$}$$

$f_i(t)$ and $\phi_j(t)$ are respectively twice and three times continuously differentiable functions which in addition are assumed to satisfy the compatibility conditions

$$\phi_1(0) = f_1(0); \quad \phi_3(1) = f_3(0); \tag{2.4.1$_6$}$$
$$\phi_1(l_1) = \phi_2(l_1) = 0; \quad \phi_2(l_2) = \phi_3(l_2) = 0.$$

It is clear that in this case the existence of two distinct phases is guaranteed only on a definite finite interval of time $(0, t^*)$. At the time t^* the domain containing the second phase degenerates to a point and for $t > t^*$ the problem reduces to a simple first boundary problem for the heat equation with one phase of constant composition

$$L_1 u = 0; \quad u(0,t) = f_1(t); \quad u(1,t) = f_3(t); \quad t > t^*, \tag{2.4.2}$$
$$u(x,t^*) = \begin{cases} u_1(x,t^*) & \text{for } 0 < x < y_1(t^*) \\ u_3(x,t^*) & \text{for } y_1(t^*) < x < 1. \end{cases}$$

The existence of some such t^* can be proved without difficulty by applying Lemma 1 to problem (2.4.1$_i$) $(i = 1, \cdots, 5)$.

We will show that despite the degeneration of the second phase domain to a point, the solution of problem (2.4.1$_i$) can be constructed in the interval $(0, t^*)$, and moreover the point $(y_1(t^*), t^*)$ is regular in the sense that along any path situated in the domain of definition of u_i and tending to the point, the limiting value of u_i exists and is equal to zero.

This last property is obvious. In fact, as was shown by I. G. Petrovskiĭ [112], the boundary point $x = y(t)$ is regular for $t = t_0$ if in a left neighborhood of $t = t_0$ the boundary $x = y(t)$ satisfies the condition

$$|y(t) - y(t_0)|^2 \leqq 4(t_0 - t) \ln \frac{1}{t_0 - t}. \tag{2.4.3}$$

In our case this automatically occurs. We first examine $y_1(t)$. Choose an arbitrary $t_0 < t^*$ and assume for definiteness that $y_1(t)$ is not decreasing at the time $t = t_0$. Let

$$M = \max\left\{ \max_{0<t<\infty} |f_i(t)|, \quad i = 1,3; \quad \max_{0<x<1;\, i=1,2,3} |\phi_i(x)| \right\} \tag{2.4.4}$$

and

$$w = -M\left[1 - \frac{x - y_1(t_0)}{y_1(t) - y_1(t_0)}\right]. \tag{2.4.5}$$

Then in some right neighborhood of the point t_0

$$L_1 w = -\frac{\partial w}{\partial t} \geqq 0. \tag{2.4.6}$$

Hence from (2.4.4) and the assumptions of the problem it follows, by virtue of the maximum principle, that in this neighborhood of t_0

$$u_1 \geqq w \text{ for } y_1(t_0) < x < y_1(t). \tag{2.4.7$_1$}$$

But by virtue of the equation

$$u_1 = w = 0 \text{ for } x = y_1(t), \tag{2.4.7$_2$}$$

this shows that

$$\frac{\partial}{\partial x} u_1 \Big|_{x=y_1(t)} \leqq \frac{M}{y_1(t) - y_1(t_0)}. \tag{2.4.7$_3$}$$

Since by the maximum principle we have

$$\frac{\partial}{\partial x} u_2 \Big|_{x=y_1(t)} \geqq 0, \tag{2.4.7$_4$}$$

we can infer from the Stefan condition (2.4.1$_4$) that

$$\dot{y}(t) \leqq \frac{M}{y_1(t) - y_1(t_0)}. \tag{2.4.8}$$

But this shows that in a right neighborhood of $t = t_0$

$$|y_1(t) - y_1(t_0)|^2 < 2M|t - t_0|. \tag{2.4.9}$$

In an analogous manner we can prove the validity of (2.4.3) for a non-increasing function $y_1(t)$ at the time t_0. Since M is defined independently of t_0 we see that $y_1(t)$ satisfies (2.4.3) in any left neighborhood of $t = t^*$.

In order to show the possibility of constructing the solution to the problem on every interval $(0, t^*)$, it suffices to note the following.

The solution to the problem (2.4.1$_i$) can be constructed in the small by reduction to a system of integral equations of the form (2.2.3$_i$) ($i = 1, 2, 3$) and applying the method of Picard or "polygonal" approximations described in Chapter I, §6. The length T of the time interval on which the sequence of approximations automatically converges depends on

$$\rho(T) = \min_{0<t\leqq T} \{\min y(t);\ \min[y_2(t) - y_1(t)];\ \min[1 - y_2(t)]\} \tag{2.4.10}$$

and tends to zero with $\rho(T)$.

Let t_0 be arbitrary but such that $\rho(t_0) > 0$. Then the solution can be constructed on the interval $(0, t_0)$ by means of continuation. Since Lemma 1 remains valid, together with the results of §2, we see that the

solution can be constructed on the interval $(0, t_0)$ in a finite number of steps.

Assume that t_0 is a lower bound of the set of all t which cannot be attained in a finite number of steps. Then it is obvious that $t_0 = t^*$, i.e. that

$$y_1(t_0) = y_2(t_0); \quad y_1(t) < y_2(t) \text{ for } t < t_0. \tag{2.4.11}$$

In fact, $t_0 < t^*$ is impossible, since the assumption that t_0 is unattainable in a finite number of steps contradicts the condition $\rho(t_0) > 0$. In exactly the same way we find that $t_0 > t^*$ is impossible, since from the assumption that t^* is attainable in a finite number of steps we infer the attainability in a finite number of steps of any $t_0 > t^*$. In fact, noting the construction of the solution of the problem (2.4.2) on an interval (t^*, t_0), we see that it is sufficient to reduce the problem to a system of integral equations of Volterra type and to solve it by means of successive approximations. This process converges for similar systems on an arbitrary finite interval of time. But this shows that the determination of the solution on an interval (t^*, t_0) requires only one step, no matter what t_0 is. Thus t^* can be attained by means of a finite or countable number of steps. Since $y_i(t)$ converges for $t \to t^*$ to $l^* \in (0, 1)$ and by our proof the point (l^*, t^*) is regular, we find that $u_i(x, t)$ tends to $u_i(x, t^*)$ uniformly in x everywhere in its domain of definition. But this completes the proof of our assertion.

§5. The third boundary problem. Existence of the solution in the large

We now consider the problem B_3:

$$\begin{aligned}
&L_1 u_1 = 0, \quad (x,t) \in D_1(y, t_0, 0, l);\\
&L_2 u_2 = 0, \quad (x,t) \in D_2(y, t_0, 1, l);\\
&u_1 = \phi_1(x), \quad 0 < x < l; \qquad u_2 = \phi_2(x), \quad l < x < 1 \text{ for } t = t_0;\\
&\left(\frac{\partial}{\partial x} - h\right) u_1 = -hf_1(t) \text{ for } x = 0; \qquad u_2|_{x=1} = f_2(t), \quad t > t_0;\\
&u_1 = u_2 = 0 \text{ for } x = y(t), \quad t > t_0;\\
&\dot{y}(t) = v_1(t) - v_2(t), \quad t > 0, \quad y(0) = l \in (0,1)\\
&\qquad\qquad \left(v_i(t) = \frac{\partial}{\partial x}\, u_i(y(t), t)\right),
\end{aligned} \tag{2.5.1}$$

assuming that f_i and ϕ_i are respectively twice and thrice continuously

differentiable functions everywhere in their domains of definition, and that they satisfy the coupling conditions

$$f_1(t_0) = \phi_1(0); \quad f_2(t_0) = \phi_2(1); \quad \phi_1(l) = \phi_2(l) = 0. \tag{2.5.2}$$

Substituting g_3 for E in the fundamental formula (1.2.2) for the evaluation of u_1, we find that

$$\begin{aligned} u_1(x,t) &= h\int_{t_0}^{t} f_1(\tau)\, g_3(x,0,t-\tau)\, d\tau \\ &\quad + \int_0^l \phi_1(\xi)\, g_3(x,\xi,t-t_0)\, d\xi + \int_{t_0}^{t} v_1(\tau)\, g_3(x,y(\tau),t-\tau)\, d\tau. \end{aligned} \tag{2.5.3}$$

Differentiating with respect to x for $t > t_0$, $0 < x < y(t)$, using the conjugacy of g_3 and g_4[5] and taking (2.5.1) and (2.5.2) into account, we find that

$$\begin{aligned} \frac{\partial}{\partial x} u_1(x,t) &= 2\phi_1(0)h\int_0^{\infty} \exp(-h\zeta)\, E(x+\zeta, t-t_0)\, d\zeta \\ &\quad + \int_0^l \dot{\phi}_1(\xi)\, g_4(x,\xi,t-t_0)\, d\xi \\ &\quad - h\int_{t_0}^{t} f_1(\tau)\frac{\partial}{\partial \xi} g_4(x,0,t-\tau)\, d\tau \\ &\quad - \int_{t_0}^{t} v_1(\tau)\frac{\partial}{\partial \xi} g_4(x,y(\tau),t-\tau)\, d\tau. \end{aligned} \tag{2.5.4}$$

Hence in the limit for $x \to y(t) - 0$

$$\begin{aligned} v_1(t) &= 4h\phi_1(0)\int_0^{\infty} \exp(-h\zeta)\, E(y(t)+\zeta, t-t_0)\, d\zeta \\ &\quad + 2\int_0^l \dot{\phi}_1(\xi)\, g_4(y(t),\xi,t-t_0)\, d\xi \\ &\quad - 2h\int_{t_0}^{t} f_1(\tau)\frac{\partial}{\partial \xi} g_4(y(t),0,t-\tau)\, d\tau \\ &\quad - 2\int_{t_0}^{t} v_1(\tau)\frac{\partial}{\partial \xi} g_4(y(t),y(\tau),t-\tau)\, d\tau. \end{aligned} \tag{2.5.5}$$

Thus in the case of problem B_3 equations $(2.2.3_1)$, $(2.2.4_1)$ and $(2.2.6_1)$ are replaced by equations (2.5.5), (2.5.3) and (2.5.4) respectively. The

[5] See the Introduction, §2.

other equations remain unchanged. Obviously every assertion of Theorems 5 and 6 remains valid.

We now suppose in addition that

$$(2.5.6_1) \qquad hm = h\inf|f_1(t)| > M = \sup|f_2(t)| \quad \text{for } t_0 < t < \infty$$

and that there exists some $l^* \in (0,1)$ such that

$$(2.5.6_2) \qquad \frac{M(x-l^*)}{1-l^*} \geqq \phi_2(x) \quad \text{for } l \leqq x \leqq 1.$$

We again define by steps a process of constructing the solution on the basis of Theorems 5 and 6. With the assumptions $(2.5.6_i)$, we obtain

THEOREM 8. a) *The solution to problem* B_3 *can be constructed on any preassigned interval of time* (t_0, T) *in a finite number of steps.*

b) *If the limit of* $f_i(t)$ *exists*:

$$(2.5.7) \qquad \lim_{t\to\infty} f_1(t) = -T_1 < 0; \quad \lim_{t\to\infty} f_2(t) = T_2 > 0,$$

then as $t \to \infty$ *the functions* $u_i(x,t)$ *and* $y(t)$ *tend to limits coinciding with the solution of the corresponding stationary problem*:

$$(2.5.8) \qquad \lim_{t\to\infty} u_1(x,t) = U_1(x) = -\frac{hT_1(L-x)}{1+Lh};$$

$$(2.5.9) \qquad \lim_{t\to\infty} u_2(x,t) = U_2(x) = T_2\frac{x-L}{1-L};$$

$$(2.5.10) \qquad \lim_{t\to\infty} y(t) = L = \frac{hT_1 - T_2}{h(T_1+T_2)}.$$

REMARK. The convergence to the limit in (2.5.8) and (2.5.9) is uniform.

We will first verify the first part of the theorem, which is a consequence of the following lemmas.

LEMMA 3. *Let* $u(x,t)$ *be superparabolic (subparabolic) in* $D = \{X_1(\tau) < x < X_2(\tau);\ t_0 < \tau \leqq t\}$ *and satisfy the conditions*

$$(2.5.11) \qquad \begin{aligned} &\left(\frac{\partial}{\partial x} - h\right) u\Big|_{x=X_1(t)} \leqq 0 \ (\geqq 0) \ \textit{for } t > t_0; \\ &u|_{x=X_2(t)} \geqq 0 \ (\leqq 0) \ \textit{for } t > t_0; \\ &u|_{t=t_0} \geqq 0 \ (\leqq 0) \ \textit{for } X_1(0) < x < X_2(0). \end{aligned}$$

Then throughout D

$$u(x,t) \geqq 0 \ (\leqq 0). \tag{2.5.12}$$

LEMMA 4. *Lemma* 1 *remains valid for the solution of problem* B_3 *if the condition* $x = 0$ *is replaced by the conditions*

$$\begin{aligned} &\left(\frac{\partial}{\partial x} - h\right) u_{1,1}\Big|_{x=0} \leqq -hf_1(t) \ \text{ for } \ t > t_0; \\ &\left(\frac{\partial}{\partial x} - h\right) u_{1,2}\Big|_{x=0} \geqq -hf_1(t) \ \text{ for } \ t > t_0. \end{aligned} \tag{2.5.13}$$

In fact, let us assume that Lemmas 3 and 4 are valid. On the basis of this assumption we will prove that the first part of the theorem is valid. With this aim we note that, as in the proof of analogous assertions for problem B_1, it suffices to prove that

$$\min\{\inf y(t), \inf_{t_0<t<\infty} (1 - y(t))\} > 0 \tag{2.5.14}$$

and

$$\sup_{t_0<t<\infty} \left|\frac{\partial}{\partial x} u_1(x,t)\right|; \quad \sup_{t_0<t<\infty} \left|\frac{\partial}{\partial x} u_2(x,t)\right| < \infty, \tag{2.5.15}$$

whence it will follow that the length of every step in the determination of the solution is independent of the length of the time interval on which the solution is already defined. An exact repetition of the reasoning used in our investigation of the first boundary problem implies that to prove (2.5.15) it suffices, assuming (2.5.14), to prove that

$$\sup_{t_0<t<\infty} \left|\frac{\partial}{\partial x} u_1(0,t)\right|; \quad \sup_{t_0<t<\infty} |u_1(0,t)|; \quad \sup_{t_0<t<\infty} \left|\frac{\partial}{\partial x} u_2(1,t)\right| < \infty. \tag{2.5.16}$$

Thus we will prove the validity of (2.5.16) under the assumption that (2.5.14) is valid. We note, first of all, that by virtue of Lemma 3

$$u_1(x,t) \leqq 0. \tag{2.5.17}$$

Hence from the boundary condition of the problem for $x = 0$ it follows that

$$\frac{\partial}{\partial x} u_1\Big|_{x=0} \leqq -hf_1(t). \tag{2.5.18}$$

On the other hand, from the nonnegativity of $v_1(t) = \partial u_1(y(t),t)/\partial x$, arising from (2.5.17), (2.5.1) and the maximum principle, from the inequality

$$\frac{\partial}{\partial \xi} g_4\Big|_{x=0} = 2h \int_0^\infty \exp(-h\zeta) \frac{\partial}{\partial \xi} E(\xi + \zeta, t - \tau)\, d\zeta \leqq 0 \quad \text{for } \xi \geqq 0 \tag{2.5.19}$$

and from (2.5.4), it follows that

$$\begin{aligned} \frac{\partial}{\partial x} u_1(0,t) > {} & 2h\phi_1(0) \int_0^\infty \exp(-h\zeta) E(\zeta, t - t_0)\, d\zeta \\ & + \int_0^l \dot{\phi}_1(\xi) g_4(0, \xi, t - t_0)\, d\xi \\ & - h \int_{t_0}^t f_1(\tau) \frac{\partial}{\partial \xi} g_4(0, 0, t - \tau)\, d\tau, \end{aligned} \tag{2.5.20}$$

i.e. that $\partial u_1(0,t)/\partial x$ is bounded below independently of $v_1(t)$ and of $y(t)$, and consequently of the length of the interval on which the solution has been constructed. This result, together with (2.5.18), yields the first of the estimates (2.5.16). Hence from the condition

$$\frac{\partial}{\partial x} u_1\Big|_{x=0} = -h(f_1(t) - u_1(0,t)) \tag{2.5.21}$$

we at once obtain the second of these estimates. Finally, the third of the estimates (2.5.16) remains the same as in the theorem for problem B_1.

Thus it remains for us to prove the validity of the estimate (2.5.14). For this purpose it suffices, since by assumption Lemma 4 is valid, to prove the existence of curves $z_j(t)$ $(j = 1, 2)$ satisfying these conditions for $t_0 \leqq t \leqq T$. With this aim we examine the functions

$$u_{1,1}(x,t) = -\frac{hm(l_1 - x)}{1 + l_1 h}; \quad u_{2,1} = \frac{M(x - l_1)}{1 - l_1}; \quad z_1(t) \equiv l_1, \tag{2.5.22}$$

where m and M are defined in agreement with $(2.5.6_1)$; we subject l_1 to the conditions

$$\frac{hml_1}{1 + l_1 h} < |\phi_1(x)| \text{ for } 0 \leqq x \leqq l_1; \quad \frac{hm}{1 + l_1 h} > \frac{M}{1 - l_1}, \tag{2.5.23}$$

as is possible by virtue of $(2.5.6_1)$, $(2.5.6_2)$ and the fact that $\phi_1(0) = f_1(t_0) < 0$. It is obvious that $u_{i,1}$ and z, as defined, satisfy the conditions of Lemma 4. Thus

$$y(t) > l_1 \text{ for } t_0 \leqq t < \infty. \tag{2.5.24}$$

Analogously, we set

$$z_2(t) \equiv l_2; \quad u_{1,2} = -\frac{hm^*(l_2 - x)}{1 + l_2 h}; \quad u_{2,2} = \frac{M^*(x - l_2)}{1 - l_2} \tag{2.5.25}$$

and subject l_2, m^* and M^* to the conditions

$$\begin{aligned} &m^* \geqq \sup|f_1(t)|; \quad M^* \leqq \inf f_2(t) \text{ for } t_0 \leqq t < \infty; \\ &\frac{hm^*(l_2 - x)}{1 + l_2 h} > |\phi_1(x)| \text{ for } 0 \leqq x \leqq l; \\ &\frac{M^*(x - l_2)}{1 - l_2} \leqq \phi_2(x) \text{ for } l_2 \leqq x \leqq 1; \\ &\frac{hm^*}{1 + l_2 h} < \frac{M^*}{1 - l_2}, \end{aligned} \tag{2.5.26}$$

as is possible since $\phi_2(1) = f_2(t_0) > 0$ and $\inf f_2(t) > 0$. It is obvious that $u_{i,2}$ $(i = 1, 2)$ and z_2 satisfy every condition of the second part of Lemma 4. Consequently

$$y(t) < l_2 \text{ for } t_0 \leqq t < \infty, \tag{2.5.27}$$

which completes our proof of the assertion under the assumption of the validity of Lemmas 3 and 4.

Lemma 4 is proved exactly as was Lemma 1, with the help of Lemma 3. We therefore omit its proof.

Lemma 3 is an obvious consequence of the maximum principle. In fact, from the maximum principle we know that $u(x, t)$ assumes its maximum on the boundary of D. Let

$$\min_{(x,t) \in \bar{D}} u(x, t) = m < 0. \tag{2.5.28}$$

By the assumptions of the problem

$$\begin{aligned} &u(x, t) \geqq 0 \text{ for } x = X_2(t), \quad t > t_0 \\ &\text{and for } X_1(t_0) < x < X_2(t_0), \; t = t_0. \end{aligned} \tag{2.5.29}$$

Consequently the value m is attained at some point

$$x = X_1(\tau); \quad t = \tau, \; \tau > t_0. \tag{2.5.30}$$

This shows that

$$\frac{\partial}{\partial x} u(X_1(\tau), \tau) \leqq h u(X_1(\tau), \tau) = hm < 0 \tag{2.5.31}$$

and consequently there is a point within D at which u achieves a smaller

value than m. But this contradiction implies the truth of Lemma 3 and at the same time the first part of Theorem 8.

REMARK. Condition $(2.5.6_1)$ is extremely stringent. Nevertheless it is "almost necessary." In fact, in the stationary state, from $L>0$ it follows that $hT_1 > T_2$, which shows that for $f_2(t)$ monotone increasing and $f_1(t)$ monotone decreasing in absolute value, we have

$$h \inf |f_1(t)| > \sup |f_2(t)|.$$

On the other hand, it is obvious that for given f_1 and for f_2 arbitrarily small it is possible to select an initial value ϕ_2 such that the domain occupied by the first phase vanishes, degenerating to a point for $t > t_0$. Thus certain limitations on $\phi_2(x)$ are necessary. However, it is obvious that the condition $(2.5.6_2)$ can be somewhat relaxed at the cost of complicating the proof.

§6. The third boundary problem. Asymptotic behavior of the solution

We turn to the proof of the second part of Theorem 8. As in the proof of Theorem 7, it suffices to show that for any $\epsilon > 0$ there exist $t_0^* > t_0$ and $t_1 \geqq t_0^*$ such that for any $t_2 \geqq t_1$ one can find functions $z_j(t)$ and $u_{i,j}(x,t)$ $(i,j=1,2)$, satisfying every requirement of Lemma 4 for $t_0^* < t < t_2$, and in addition such that

$$z_1(t_0^*) = 0; \quad z_1(t_2) = L - \epsilon; \quad z_2(t_0^*) = 0; \quad z_2(t_2) = L + \epsilon; \tag{2.6.1}$$

here L is defined in agreement with (2.5.10), since using this proof the theorem can be extended as was Theorem 7 in §3.

For simplicity we choose $t_0^* = t_0 = 0$.

The existence of the required functions $u_{i,j}$ and z_j is a consequence of the following lemmas.

LEMMA 5. *Let $\eta > 0$ be arbitrarily small and let*

$$T_{1,1} = T_1 - \eta; \quad T_{1,2} = T_1 + \eta; \quad T_{2,1} = T_2 + \eta; \quad T_{2,2} = T_2 - \eta. \tag{2.6.2}$$

Then there exists some $t_0 > 0$ so large that

$$0 < T_{1,1} < |f_1(t)| < T_{1,2}; \quad 0 < T_{2,2} < f_2(t) < T_{2,1} \text{ for } t \geqq t_0 \tag{2.6.3}$$

and

$$\begin{aligned} u_1(x,t_0) &\geqq -h\frac{T_{1,2}(1-x)}{1+h} \text{ for } 0 < x < y(t_0); \\ u_2(x,t_0) &\leqq T_{2,1}x \text{ for } y(t_0) < x < 1. \end{aligned} \tag{2.6.4}$$

LEMMA 6. *Let q be an arbitrary positive constant. Let w_1 be uniformly bounded throughout its domain of definition,*

$$L_1 w_1 = 0 \text{ for } 0 < x < 1,\ t > 0;$$

$$\left(\frac{\partial}{\partial x} - h\right) w_1\Big|_{x=0} = hF_1(t); \quad w_1|_{x=1} = 0; \quad w_1|_{t=0} = \psi(x), \tag{2.6.5}$$

where ψ and F_1 satisfy the following conditions

$$\psi(x) \text{ is continuous for } 0 \leqq x \leqq 1;$$

$$\psi(x) < 0 \text{ for } 0 < x < 1;$$

$$\psi(x) \leqq -\max|u_1(x,0) - q| \text{ for } 0 < x < y(0); \tag{2.6.6}$$

$$-F_1(t) \leqq f_1(t) - \frac{\eta}{2} < 0; \quad \lim_{t\to\infty} F_1(t) = T_{1,2}.$$

Then

$$\lim w_1 = -\frac{hT_{1,2}}{1+h}(1-x) \tag{2.6.7}$$

and in addition

$$\inf_{0<t<\infty} |w_1(y(t),t)| = \mu > 0. \tag{2.6.8}$$

LEMMA 7. *Let q be an arbitrary positive constant and let*

$$L_2 w_2 = 0 \text{ for } 0 < x < 1,\ t > 0;$$

$$w_2|_{x=0} = 0; \quad w_2|_{t=t_1} = \psi(x); \quad w_2|_{x=1} = F(t), \tag{2.6.5*}$$

where

$$\psi(x) > 0 \text{ for } 0 < x < y(t_1);$$

$$\psi(x) > \max(u_2 + q) \text{ for } y(t_1) < x < 1;$$

$$\psi(x) \text{ is continuous for } 0 < x < 1; \tag{2.6.6*}$$

$$F(t) \geqq f_2(t) + \frac{\eta}{2} \text{ for } t \geqq t_1, \ \lim_{t\to\infty} F(t) = T_{2,1}.$$

Then

$$\lim_{t\to\infty} w_2 = T_{2,1}x, \tag{2.6.7*}$$

and moreover

$$\inf_{0<t<\infty} |w_2(y(t),t)| > 0. \tag{2.6.8*}$$

Let us assume that Lemma 5 is proved. For simplicity, set $t_0 = 0$. We

will then prove the existence of numbers $\alpha_1^* > 0$ and $\alpha_2^* > 0$ so small that if

$$\alpha_1^* t_1 = L - \epsilon; \quad 1 - \alpha_2^* t_1 = L + \epsilon \tag{2.6.9}$$

and $t_2 > t_1$ are arbitrarily large, then the functions $u_{i,j}(x,t)$, $z_j(t)$ determined by the conditions

$$\begin{aligned}
&L_1 u_{1,1} = 0 \text{ for } 0 < x < z_1(t);\\
&\left(\frac{\partial}{\partial x} - h\right) u_{1,1}\Big|_{x=0} = hT_{1,1}; \qquad u_{1,1}|_{x=z_1(t)} = 0;\\
&L_2 u_{2,1} = 0 \text{ for } z_1(t) < x < 1; \qquad u_{2,1}|_{x=1} = T_{2,1};\\
&u_{2,1}|_{x=z_1(t)} = 0; \qquad u_{2,1}|_{t=0} = T_{2,1}x;\\
&z_1(t) = \alpha_1 t; \qquad \alpha_1 t_2 = L - \epsilon; \qquad 0 \leqq t \leqq t_2;
\end{aligned} \tag{2.6.10$_1$}$$

$$\begin{aligned}
&L_1 u_{1,2} = 0 \text{ for } 0 < x < z_2(t); \quad \left(\frac{\partial}{\partial x} - h\right) u_{1,2}\Big|_{x=0} = hT_{1,2};\\
&u_{1,2}|_{x=z_2(t)} = 0; \quad u_{1,2}|_{t=0} = \frac{-hT_{1,2}(1-x)}{1+h};\\
&L_2 u_{2,2} = 0 \text{ for } z_2(t) < x < 1; \quad u_{2,2}|_{x=z_2(t)} = 0; \quad u_{2,2}|_{x=1} = T_{2,2};\\
&z_2(t) = 1 - \alpha_2 t; \quad 1 - \alpha_2 t_2 = L + \epsilon, \ 0 \leqq t \leqq t_2
\end{aligned} \tag{2.6.10$_2$}$$

satisfy every condition of Lemma 5 for arbitrary

$$\alpha_1 < \alpha_1^*; \qquad \alpha_2 < \alpha_2^*. \tag{2.6.10$_3$}$$

Obviously it is sufficient to prove that

$$\begin{aligned}
\alpha_1 = \dot{z}_1 &< \left(\frac{\partial}{\partial x} u_{1,1} - \frac{\partial}{\partial x} u_{2,1}\right)\Big|_{x=z_1=\alpha_1 t}, \quad 0 < t \leqq t_2;\\
-\alpha_2 = \dot{z}_2 &> \left(\frac{\partial}{\partial x} u_{1,2} - \frac{\partial}{\partial x} u_{2,2}\right)\Big|_{x=z_2=1-\alpha_2 t}, \quad 0 < t \leqq t_2.
\end{aligned} \tag{2.6.11}$$

We begin by examining $u_{2,1}(x,t)$. Set

$$u_{2,1}(x,t) = T_{2,1}\frac{x - \alpha_1 t}{1 - \alpha_1 t} + w_{2,1}(x,t). \tag{2.6.12}$$

By virtue of (2.6.10$_1$) we have

$$\begin{aligned}
&L_2 w_{2,1} = -\alpha_1 T_{2,1}\frac{1-x}{(1-\alpha_1 t)^2} \text{ for } \alpha_1 t < x < 1;\\
&w_{2,1}|_{t=0} = 0; \quad w_{2,1}|_{x=\alpha_1 t} = 0; \quad w_{2,1}|_{x=1} = 0.
\end{aligned} \tag{2.6.13}$$

Let $G(x, \xi, t, \tau)$ be the Green's function for the first boundary problem of the equation $L_2 u = 0$ on the interval $[\alpha_1 t, 1]$. Then

$$w_{2,1} = \alpha_1 T_{2,1} \int_0^t d\tau \int_{\alpha_1 \tau}^1 \frac{1-\xi}{(1-\alpha_1 \tau)^2} G(x, \xi, t, \tau)\, d\xi, \tag{2.6.14}$$

whence

$$\frac{\partial}{\partial x} w_{2,1} \Big|_{x=\alpha_1 t} = \alpha_1 T_{2,1} \int_0^t d\tau \int_{\alpha_1 \tau}^1 \frac{1-\xi}{(1-\alpha_1 \tau)^2} \frac{\partial}{\partial x} G(x, \xi, t, \tau)\, d\xi. \tag{2.6.15}$$

Now let

$$g^*(x, \xi, t, \tau) = E(x-\xi, a^2(t-\tau)) \left[1 - \exp\left(- \frac{(x-\alpha_1 t)(\xi - \alpha_1 \tau)}{a^2 (t-\tau)} \right) \right] \tag{2.6.16}$$

be the Green's function of the first boundary problem for the equation $L_2 u = 0$ on the halfline $x > \alpha_1 t$, and

$$K(x, \xi, t, \tau) = g^*(x, \xi, t, \tau) - G(x, \xi, t, \tau). \tag{2.6.17}$$

Considered as a function of x and t, K satisfies the conditions

$$L_2 K = 0; \quad K|_{x=1} = g^*|_{x=1} > 0; \quad K|_{t=\tau+0} = 0; \quad K|_{x=\alpha_1 t} = 0. \tag{2.6.18}$$

Consequently, by virtue of the maximum principle,

$$K > 0, \ \alpha_1 t < x < 1, \ t > \tau, \tag{2.6.19}$$

whence from the equation

$$g^*|_{x=\alpha_1 t} = G|_{x=\alpha_1 t} = 0$$

it follows that

$$\frac{\partial}{\partial x} K \Big|_{x=\alpha_1 t} = \frac{\partial}{\partial x} g^* \Big|_{x=\alpha_1 t} - \frac{\partial}{\partial x} G \Big|_{x=\alpha_1 t} \geqq 0. \tag{2.6.20}$$

Inserting this estimate into (2.6.15), we find that

$$\frac{\partial}{\partial x} w_{2,1} \Big|_{x=z_1} \leqq \alpha_1 T_{2,1} \int_0^t d\tau \int_{\alpha_1 \tau}^1 \frac{1-\xi}{(1-\alpha_1 \tau)^2} \frac{\partial}{\partial x} g^*(x, \xi, t, \tau) \Big|_{x=\alpha_1 t} d\xi = I. \tag{2.6.21}$$

Using (2.6.16) we see that

$$I=\frac{\alpha_1 T_{2,1}}{2a^3\sqrt{\pi}}\Big[\int_0^t d\tau\int_{\alpha_1\tau}^{\alpha_1 t}+\int_0^t d\tau\int_{\alpha_1 t}^1 \frac{1-\xi}{(1-\alpha_1\tau)^2}\frac{\xi-\alpha_1\tau}{(t-\tau)^{3/2}} \cdot\exp\left(-\frac{(\alpha_1 t-\xi)^2}{4a^2(t-\tau)}\right)d\xi\Big]=I'+I''. \tag{2.6.22}$$

We have

$$I'=\frac{\alpha_1 T_{2,1}}{2a^3\sqrt{\pi}}\int_0^t d\tau\int_{\alpha_1\tau}^{\alpha_1 t}\frac{1-\xi}{(1-\alpha_1\tau)^2}\frac{\xi-\alpha_1\tau}{(t-\tau)^{3/2}} \cdot\exp\left[-\frac{(\alpha_1 t-\xi)^2}{4a^2(t-\tau)}\right]d\xi<\frac{\alpha_1^3 T_{2,1}t^{3/2}}{3a^3\sqrt{\pi}(1-\alpha_1 t)}. \tag{2.6.23}$$

Consequently for $0\leqq t\leqq t_2<(L-\epsilon)/\alpha_1$ we have

$$0<I'<\frac{\alpha_1^{3/2}T_{2,1}(L-\epsilon)^{3/2}}{3a^3\sqrt{\pi}(1+\epsilon-L)}. \tag{2.6.24}$$

Moreover,

$$I''=\frac{\alpha_1 T_{2,1}}{2a^3\sqrt{\pi}}\int_0^t d\tau\int_{\alpha_1 t}^1\frac{1-\xi}{(1-\alpha_1\tau)^2}\frac{\xi-\alpha_1\tau}{(t-\tau)^{3/2}}\exp\left[-\frac{(\alpha_1 t-\xi)^2}{4a^2(t-\tau)}\right]d\xi =\dot{I}_1+\dot{I}_2. \tag{2.6.25}$$

where

$$\dot{I}_1=\frac{\alpha_1 T_{2,1}}{2a^3\sqrt{\pi}}\int_0^t d\tau\int_{\alpha_1 t}^1\frac{1-\xi}{(1-\alpha_1\tau)^2}\frac{\xi-\alpha_1 t}{(t-\tau)^{3/2}}\exp\left[-\frac{(\alpha_1 t-\xi)^2}{4a^2(t-\tau)}\right]d\xi;$$
$$\dot{I}_2=\frac{\alpha_1^2 T_{2,1}}{2a^3\sqrt{\pi}}\int_0^t d\tau\int_{\alpha_1 t}^1\frac{1-\xi}{(1-\alpha_1\tau)^2}\frac{1}{\sqrt{t-\tau}}\exp\left[-\frac{(\alpha_1 t-\xi)^2}{4a^2(t-\tau)}\right]d\xi. \tag{2.6.26}$$

Obviously

$$0<\dot{I}_2<\frac{\alpha_1^2 T_{2,1}\sqrt{t}}{2a^3\sqrt{\pi}}<\frac{\alpha_1^{3/2}T_{2,1}(L-\epsilon)^{1/2}}{2a^3\sqrt{\pi}}; \tag{2.6.27$_1$}$$

$$0<\dot{I}_1<\frac{\alpha_1 T_{2,1}}{2a^3\sqrt{\pi}}\int_{\alpha_1 t}^1\frac{1-\xi}{(1-\alpha_1 t)^2}d\xi\int_0^t\frac{\xi-\alpha_1 t}{(t-\tau)^{3/2}}\exp\left[-\frac{(\alpha_1 t-\xi)^2}{4a^2(t-\tau)}\right]d\tau$$
$$=\frac{\alpha_1 T_{2,1}}{a^2(1-\alpha_1 t)^2}\int_{\alpha_1 t}^1(1-\xi)\operatorname{erfc}\frac{\xi-\alpha_1 t}{2a\sqrt{t}}d\xi<\frac{\alpha_1 T_{2,1}}{2a^2}. \tag{2.6.27$_2$}$$

Substituting (2.6.27$_i$) ($i=1,2$), (2.6.25) and (2.6.24) into (2.6.21), we find that

$$0<\frac{\partial w_{2,1}}{\partial x}\Big|_{x=\alpha_1 t}<A\alpha_1 \text{ for } 0\leqq t\leqq t_2, \tag{2.6.28}$$

where $A = \text{const}$ is independent of t. Hence from (2.6.12) we see that

$$\left.\frac{\partial u_{2,1}}{\partial x}\right|_{x=z_1=\alpha_1 t} = \frac{T_{2,1}}{1-\alpha_1 t} + v_{2,1}, \tag{2.6.29}$$

where

$$0 < v_{2,1} < A\alpha_1; \qquad 0 \leqq t \leqq t_2. \tag{2.6.29*}$$

We now construct $u_{1,1}$. It is sought in the form

$$u_{1,1}(x,t) = \int_{-\infty}^{0} \phi(\xi) g'(x,\xi,t,0)\, d\xi, \tag{2.6.30}$$

where g' is obtained from the expression (2.6.16) for g^*, replacing a^2 by 1. From (2.6.30) and (2.6.16) it follows that

$$\frac{\partial u_{1,1}}{\partial x} = \int_{-\infty}^{0} \phi(\xi) \frac{\exp(-(x-\xi)^2/4t)}{2\sqrt{\pi}t^{3/2}} \left(\frac{\xi - x}{2}\left(1 - \exp\left[-\frac{\xi}{t}(x-\alpha_1 t)\right]\right)\right. \tag{2.6.31}$$
$$\left. + \xi \exp\left[-\frac{\xi}{t}(x-\alpha_1 t)\right]\right) d\xi.$$

Hence

$$\left.\frac{\partial u_{1,1}}{\partial x}\right|_{x=\alpha_1 t} = \int_{-\infty}^{0} \phi(\xi)\xi \frac{\exp(-(\alpha_1 t - \xi)^2/4t)}{2\sqrt{\pi}t^{3/2}} d\xi \tag{2.6.32}$$

and

$$\left.\frac{\partial u_{1,1}}{\partial x}\right|_{x=0} = \int_{-\infty}^{0} \phi(\xi)(1+\exp(\alpha_1\xi)) \frac{\xi \exp(-\xi^2/4t)}{4\sqrt{\pi}t^{3/2}} d\xi. \tag{2.6.33}$$

Integrating (2.5.33) by parts and assuming that

$$\phi(0) = 0; \quad \lim_{\xi \to -\infty} \phi(\xi)(\exp(\alpha_1\xi) + 1) \frac{\exp(-\xi^2/4t)}{2\sqrt{\pi}t} = 0, \ t > 0, \tag{2.6.34}$$

we find

$$\left.\frac{\partial u_{1,1}}{\partial x}\right|_{x=0} = \int_{-\infty}^{0} [\dot{\phi}(\xi)(1+\exp(\alpha_1\xi)) + \alpha_1\phi(\xi)\exp(\alpha_1\xi)] \frac{\exp(-\xi^2/4t)}{2\sqrt{\pi t}} d\xi. \tag{2.6.35}$$

On the other hand

$$u_{1,1}|_{x=0} = \int_{-\infty}^{0} \phi(\xi)(1-\exp(\alpha_1\xi))\exp\left(-\frac{\xi^2}{4t}\right) \frac{d\theta}{2\sqrt{\pi t}}. \tag{2.6.36}$$

We now require that

$$\left(\frac{\partial}{\partial x} - h\right) u_{1,1}\Big|_{x=0} = hT_{1,1}. \tag{2.6.37}$$

Since

$$\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{0} \exp\left(-\frac{\xi^2}{4t}\right) d\xi = \frac{1}{2},$$

it follows from (2.6.35) and (2.6.36) that (2.6.37) is fulfilled if

$$\dot{\phi}(\xi)(1 + \exp(\alpha_1\xi)) + \phi(\xi)[(\alpha_1 + h)\exp(\alpha_1\xi) - h] = 2hT_{1,1}. \tag{2.6.38}$$

Integrating (2.6.38) under the initial condition $\phi(0) = 0$, we find that

$$\phi(\xi) = 2hT_{1,1}\int_0^{\xi} \exp[h(\xi - \zeta)]\left(\frac{1 + \exp(\alpha_1\zeta)}{1 + \exp(\alpha_1\xi)}\right)^{2h/\alpha_1} \frac{d\zeta}{(1 + \exp(\alpha_1\xi))}. \tag{2.6.39}$$

Substituting this into (2.6.32), we obtain

$$\frac{\partial u_{1,1}}{\partial x}\Big|_{x=\alpha_1 t} = \frac{hT_{2,1}}{\sqrt{\pi}t^3}\int_{-\infty}^{0} \frac{\xi \exp(-(\alpha_1 t - \xi)^2/4t)}{1 + \exp(\alpha_1\xi)}\, d\xi$$
$$\cdot \int_0^{\xi} \exp[h(\xi - \zeta)] \cdot \left[\frac{1 + \exp(\alpha_1\zeta)}{1 + \exp(\alpha_1\xi)}\right]^{2h/\alpha_1} d\zeta.$$

The substitution

$$\xi = -2\sqrt{tx^2}; \qquad \zeta = -2\sqrt{tz^2}$$

transforms this integral into one of the form

$$\frac{\partial u_{1,1}}{\partial x}\Big|_{x=\alpha_1 t}$$
$$= \frac{4hT_{1,1}}{\sqrt{\pi}}\int_0^{\infty} \frac{x\exp(-\alpha_1^2 t/4 - x^2)}{\operatorname{ch}(\alpha_1 x\sqrt{t})}\, dx \cdot \int_0^{x} \left(\frac{\operatorname{ch}(\alpha_1 z\sqrt{t})}{\operatorname{ch}(\alpha_1 x\sqrt{t})}\right)^{2h/\alpha_1} dz. \tag{2.6.40}$$

We now set

$$\tau = \sqrt{\alpha_1 t}, \qquad \beta = \sqrt{\alpha_1} \tag{2.6.41}$$

and pass to the limit in (2.6.40) for $\beta \to 0$ and $0 \leqq \tau \leqq \sqrt{L - \epsilon}$. Since

$$\lim_{\beta\to 0}\left(\frac{\operatorname{ch}(\alpha_1\sqrt{t}z)}{\operatorname{ch}(\alpha_1\sqrt{t}x)}\right)^{2h/\alpha_1} = \lim_{\beta\to 0}\left(\frac{\operatorname{ch}(\beta\tau z)}{\operatorname{ch}(\beta\tau x)}\right)^{2h/\beta^2}$$
$$= \exp(-h\tau^2(x^2 - z^2)), \tag{2.6.42}$$

we find that

$$\lim_{\beta\to 0}\frac{\partial u_{1,1}}{\partial x}\bigg|_{x=z_1(t)}$$

$$(2.6.43^*)\qquad = \frac{4hT_{1,1}}{\sqrt{\pi}}\int_0^\infty x\exp(-x^2)\,dx\int_0^x \exp(-h\tau^2(x^2-z^2))\,dz.$$

Changing the order of integration, we obtain

$$\lim_{\beta\to 0}\frac{\partial u_{1,1}}{\partial x}\bigg|_{x=z_1(t)}$$

$$(2.6.43)\qquad = \frac{4hT_{1,1}}{\sqrt{\pi}}\int_0^\infty \exp(h^2\tau^2z^2)\,dz\int_z^\infty \exp(-x^2(1+h^2\tau^2))\,x\,dx$$

$$= \frac{hT_{1,1}}{1+h\tau^2}.$$

But by virtue of (2.6.41) this shows that

$$(2.6.44)\qquad \frac{\partial u_{1,1}}{\partial x}\bigg|_{x=z_1(t)} = \frac{hT_{1,1}}{1+h\alpha_1 t} + v_{1,1},$$

where

$$(2.6.44^*)\qquad \lim_{\alpha_1\to 0} v_{1,1} = 0,$$

and in addition the convergence to a limit in (2.6.44*) is obviously uniform with respect to t for $0 \leqq \alpha_1 t \leqq L-\epsilon$.

From (2.6.44) and (2.6.29) it follows that

$$(2.6.45)\qquad \left(\frac{\partial u_{1,1}}{\partial x}-\frac{\partial u_{2,1}}{\partial x}\right)\bigg|_{x=\alpha_1 t} = \frac{hT_{1,1}}{1+h\alpha_1 t}-\frac{T_{2,1}}{1-\alpha_1 t}+w_1(\alpha_1 t),$$

with $w_1(\alpha_1, t)$ tending uniformly to zero for $\alpha_1\to 0$ and $0 \leqq \alpha_1 t \leqq \alpha_1 t_2 = L-\epsilon$.

Using (2.6.2) and (2.5.10), we find that for $0 \leqq \alpha_1 t \leqq \alpha_1 t_2 = L-\epsilon$

$$(2.6.46)\qquad \frac{hT_{1,1}}{1+h\alpha_1 t}-\frac{T_{2,1}}{1-\alpha_1 t} \geqq \frac{h(T_1+T_2)\epsilon-(1+h)\eta}{[1+h(L-\epsilon)](1-L+\epsilon)}.$$

Now let $\eta > 0$ satisfy the condition

$$(2.6.47)\qquad \eta < \frac{h(T_1+T_2)\epsilon}{2(1+h)}$$

and let $t_1 > 0$ be so large that if

$$(2.6.48)\qquad \alpha_1^* t_1 = L-\epsilon,$$

then for any $\alpha_1 < \alpha_1^*$

$$|w_1(\alpha_1 t)| < \frac{h(T_1+T_2)\epsilon}{4[1+h(L-\epsilon)](1-L+\epsilon)} \tag{2.6.49}$$

and in addition

$$\alpha_1^* < \frac{h(T_1+T_2)\epsilon}{4[1+h(L-\epsilon)](1-L+\epsilon)}. \tag{2.6.49*}$$

Then by virtue of (2.6.45) we find that

$$\dot{z}_1 = \alpha_1 < \left(\frac{\partial}{\partial x} u_{1,1} - \frac{\partial}{\partial x} u_{2,1}\right)\Big|_{x=z_1}, \tag{2.6.50}$$

which completes the proof.

A completely analogous proof can be given to show that the functions $u_{i,2}$ and z_2 defined by $(2.6.10_2)$ satisfy the requirement (2.6.11) for α_2^* sufficiently small, thus completing the proof of Theorem 8 under the assumption that Lemma 5 is valid.

We now prove that Lemma 5 is in fact valid if Lemmas 6 and 7 are valid. Suppose that w_1 is defined by the conditions of Lemma 6. Let

$$q^* = \min\left(\mu, q, \frac{\eta}{2}\right); \quad w^* = w_1 - u_1 \text{ for } 0 < x < y(t), \tag{2.6.51}$$

where μ, q and η are the constants appearing in the statement of Lemmas 5, 6 and 7. Then

$$L_1 w^* = 0; \quad \left(\frac{\partial}{\partial x} - h\right) w^*|_{x=0} \geqq \frac{h\eta}{2}. \tag{2.6.52}$$

It is obvious that $w^* + q^*$ satisfies every condition of Lemma 3. Consequently

$$w^* + q^* \equiv w_1(x,t) - u_1(x,t) + q^* \leqq 0 \text{ for } t \in [0, \infty). \tag{2.6.53}$$

We now choose t_0^* so large that

$$w_1(x,t) \geqq -\frac{hT_{1,2}}{1+h}(1-x) - q^* \text{ for } t \geqq t_0^*. \tag{2.6.54}$$

This can be done by virtue of Lemma 6. Inserting (2.6.54) into (2.6.53), we find that

$$u_1(x,t) \geqq -\frac{hT_{1,2}}{1+h}(1-x); \quad t \geqq t_0^*. \tag{2.6.55}$$

Thus the first of the assertions (2.6.4) of Lemma 5 is proved. The second assertion is proved with the aid of Lemma 7 in a completely analogous way. Finally, (2.6.3) stipulates the requirements of (2.1.1).

Thus it remains for us only to prove Lemmas 6 and 7. Let w_1 be defined by the conditions of Lemma 6. Choose $t_0 > 0$ such that for $t > t_0$

$$|F_1(t) - T_{1,2}| < \epsilon \tag{2.6.56}$$

where $\epsilon > 0$ is arbitrarily small. Moreover,

$$w_1 = -\frac{hT_{1,2}}{1+h}(1-x) + w_{1,1} + w_{1,2} + w_{1,3}, \tag{2.6.57}$$

where

$$L_1 w_{1,i} = 0, \quad 0 < x < 1, \quad t > t_0, \quad i = 1,2,3;$$

$$\left(\frac{\partial}{\partial x} - h\right) w_{1,1}\Big|_{x=0} = 0; \qquad w_{1,1}|_{x=1} = 0;$$

$$w_{1,1}|_{t=t_0} = w_1|_{t=t_0} + \frac{hT_{1,2}}{1+h}(1-x) \equiv w_0(x);$$

$$w_{1,2}|_{x=1} = 0; \qquad w_{1,3}|_{x=1} = 0; \qquad w_{1,2}|_{t=t_0} = 0; \qquad w_{1,3}|_{t=t_0} = 0; \tag{2.6.58}$$

$$\left(\frac{\partial}{\partial x} - h\right) w_{1,2}\Big|_{x=0} = \begin{cases} h(F_1 - T_{1,2}) \equiv \left(\dfrac{\partial}{\partial x} - h\right) w_1\Big|_{x=0}, \\ \qquad\qquad \text{if } F_1 - T_{1,2} < 0 \\ 0 \qquad\qquad \text{if } F_1 - T_{1,2} \geqq 0; \end{cases}$$

$$\left(\frac{\partial}{\partial x} - h\right) w_{1,3}\Big|_{x=0} = \begin{cases} 0 \qquad\qquad \text{if } F_1 - T_{1,2} < 0 \\ h(F_1 - T_{1,2}) \quad \text{if } F_1 - T_{1,2} \geqq 0. \end{cases}$$

Then by Lemma 3 it is obvious that for $t \geqq t_0$, $0 \leqq x \leqq 1$ we have

$$0 < w_{1,2} < \epsilon; \qquad 0 < w_{1,3} < \epsilon. \tag{2.6.59}$$

In fact, by Lemma 3 and condition (2.6.58) we have

$$w_{1,2} \geqq 0; \qquad w_{1,3} \geqq 0, \tag{2.6.60}$$

everywhere in their domain of definition. Assume in contradiction to (2.6.59) that

$$\max_{\substack{0 \leqq x \leqq 1 \\ t \geqq t_0}} w_{1,2} = M > \epsilon; \qquad \min_{\substack{0 \leqq x \leqq 1 \\ t \geqq t_0}} w_{1,3} = -m < -\epsilon. \tag{2.6.61}$$

Since $L_1 w_{1,i} = 0$, it follows that $w_{1,2}$ and $w_{1,3}$ attain the values M and

$-m$ respectively on the boundary of their domain of definition, i.e. (by condition (2.6.8)) for $x = 0$. Thus let

$$w_{1,2}(0, \tau_2) = M; \qquad w_{1,3}(0, \tau_3) = -m. \tag{2.6.62}$$

Since

$$\left(\frac{\partial}{\partial x} - h\right) w_{1,2}\Big|_{x=0} \leqq 0 \text{ and } |F_1 - T_{1,2}| < \epsilon, \tag{2.6.63}$$

we see that

$$-h\epsilon \leqq \left(\frac{\partial}{\partial x} - h\right) w_{1,2}\Big|_{x=0} \leqq 0. \tag{2.6.64}$$

Analogously

$$0 \leqq \left(\frac{\partial}{\partial x} - h\right) w_{1,3}\Big|_{x=0} < h\epsilon. \tag{2.6.64*}$$

From (2.6.62), (2.6.64), (2.6.64*) and (2.6.58) it follows that

$$\begin{aligned} 0 < h(M - \epsilon) &\leqq \frac{\partial}{\partial x} w_{1,2}\Big|_{x=0;t=\tau_2}; \\ \frac{\partial}{\partial x} w_{1,3}\Big|_{x=0;t=\tau_3} &\leqq h(\epsilon - m) < 0. \end{aligned} \tag{2.6.65}$$

But this shows that for $0 < x < 1$ there are points x_2 and x_3 such that

$$w_{1,2}(x_2, \tau_2) > M; \qquad w_{1,3}(x_3, \tau_3) < -m, \tag{2.6.66}$$

which contradicts the definition of M and m. Hence (2.6.59) is proved. Thus for every $t > t_0$

$$|w_{1,2} + w_{1,3}| < \epsilon. \tag{2.6.67}$$

On the other hand, $w_{1,1}$ admits a representation of the form

$$w_{1,1} = \int_0^1 w_0(\xi) g(x, \xi, t - t_0)\, d\xi, \tag{2.6.68}$$

where g is the Green's function for the boundary problem defining w_1. From this representation it obviously follows that

$$\lim_{t \to \infty} w_{1,1} = 0. \tag{2.6.69}$$

Comparing (2.6.67), (2.6.69) and (2.6.57), we see that assertion (2.6.7) of Lemma 6 is proved for $\epsilon > 0$ arbitrarily small.

Finally, (2.6.8) is an obvious implication of Lemma 3. In fact, from that lemma we see that $w_1 \leqq 0$ everywhere in its domain of definition. But this shows that $w_1 < 0$ for every $x \in (0,1)$, since otherwise w_1 will attain an interior maximum, and consequently by virtue of the strong maximum principle it will be identically equal to zero. We now note that, as proved above,

$$\min\left\{\inf_{0<t<\infty} y(t),\ \inf(1-y(t))\right\} = \alpha > 0. \tag{2.6.70}$$

Consequently there exist l_1 and l_2 such that

$$0 < l_1 < y(t) < l_2 < 1 \text{ for } 0 < t < \infty. \tag{2.6.71}$$

Examining w_1 in the domain $l_1 < x < l_2$, $t > 0$, we find that

$$w_1 \leqq -m_0 < 0, \tag{2.6.72}$$

which proves (2.6.8).

Lemma 7 is proved in exactly the same way. Hence we omit its proof.

§7. The Cauchy-Stefan problem

We now examine problem C, assuming that

$$\begin{aligned} &\phi_1(0) = \phi_2(0) = 0; \quad y(0) = 0; \\ &\lim_{x\to-\infty} \phi_1(x) = -\alpha_1 < 0; \quad \lim_{x\to+\infty} \phi_2(x) = \alpha_2 > 0; \\ &\lim \dot{\phi}_i(x) = 0 \text{ for } |x| \to \infty. \end{aligned} \tag{2.7.1}$$

We assume from the outset that the $\phi_i(x)$ are thrice continuously differentiable functions.

The equations $(2.2.3_i)$ $(i = 1,2,3)$, $(2.2.4_i)$ $(i = 1,2)$ and $(2.2.6_i)$ $(i = 1,2)$ are now replaced by the equations

$$\begin{aligned} v_1(t) &= 2\int_{-\infty}^{0} \dot{\phi}_1(\xi) E(y(t) - \xi, t - t_0)\, d\xi \\ &\quad - 2\int_{t_0}^{t} v_1(\tau) \frac{\partial}{\partial \xi} E(y(t) - y(\tau), t - \tau)\, d\tau; \\ v_2(t) &= 2\int_{0}^{\infty} \dot{\phi}_2(\xi) E(y(t) - \xi, a^2(t - t_0))\, d\xi \\ &\quad + 2a^2 \int_{t_0}^{t} v_2(\tau) \frac{\partial}{\partial \xi} E(y(t) - y(\tau), a^2(t - \tau))\, d\tau, \end{aligned} \tag{2.7.2}$$

$$y(t) = \int_{t_0}^{t} [v_1(\tau) - v_2(\tau)]d\tau; \tag{2.7.3}$$

$$\begin{cases} u_1(x,t) = \int_{-\infty}^{0} \phi_1(\xi) E(x-\xi, t-t_0)\, d\xi \\ \qquad + \int_{t_0}^{t} v_1(\tau) E(x-y(\tau), t-\tau)\, d\tau; \\ u_2(x,t) = \int_{0}^{\infty} \phi_2(\xi) E(x-\xi, a^2(t-t_0))\, d\xi \\ \qquad - a^2 \int_{t_0}^{t} v_2(\tau) E(x-y(\tau), a^2(t-\tau))\, d\tau, \end{cases} \tag{2.7.4}$$

$$\begin{cases} \dfrac{\partial u_1}{\partial x} = \int_{-\infty}^{0} \dot{\phi}_1(\xi) E(x-\xi, t-t_0)\, d\xi \\ \qquad - \int_{t_0}^{t} v_1(\tau) \dfrac{\partial}{\partial \xi} E(x-y(\tau), t-\tau)\, d\tau; \\ \dfrac{\partial u_2}{\partial x} = \int_{0}^{\infty} \dot{\phi}_2(\xi) E(x-\xi, a^2(t-t_0))\, d\xi \\ \qquad + a^2 \int_{t_0}^{t} v_2(\tau) \dfrac{\partial}{\partial \xi} E(x-y(\tau), a^2(t-\tau))\, d\tau. \end{cases} \tag{2.7.5}$$

Here, as everywhere above,

$$\frac{\partial}{\partial \xi} E(x-y(\tau), t) = \frac{\partial}{\partial \xi} E(x-\xi, t)\Big|_{\xi=y(\tau)}.$$

From equations (2.7.2) and (2.7.5) it follows that the theorems of existence, uniqueness and stability of the solution in the small for problems B_1 and B_3 remain valid for problem C. In addition, the length of the interval on which existence of the solution is guaranteed is limited by a value ρ with

$$\rho^{-1} = \max\left\{ \sup_{-\infty < x < 0} |\dot{\phi}_1(x)|; \quad \sup_{0 < x < \infty} |\dot{\phi}_2(x)| \right\}, \tag{2.7.6}$$

which, as above, tends to zero with it.

Consequently, in order to establish the possibility of extending the solution to a preassigned given interval of time, it is sufficient to prove the validity of the a priori estimate

$$\left|\frac{\partial}{\partial x} u_i(x,t)\right| \leqq N = N(T) \text{ for } t_0 \leqq t \leqq T, \tag{2.7.7}$$

where $N(T)$ is some constant independent of the length of the time interval on which the solution to the problem is actually constructed. But the validity of this estimate is obvious.

In fact, let

$$U_1 = u_1 + \alpha_1; \quad U_2 = u_2 - \alpha_2. \tag{2.7.8}$$

Then U_1 and U_2 satisfy the conditions

$$\begin{aligned} &L_1 U_1 = 0, \ -\infty < x < y(t); \quad L_2 U_2 = 0, \quad y(t) < x < \infty, \quad t > t_0; \\ &U_1|_{t=t_0} = \phi_1(x) + \alpha_1; \quad U_2|_{t=t_0} = \phi_2(x) - \alpha_2; \\ &U_1|_{x=y(t)} = \alpha_1; \quad U_2|_{x=y(t)} = -\alpha_2; \\ &\lim_{x \to -\infty} U_1 = 0; \quad \lim_{x \to +\infty} U_2 = 0, \end{aligned} \tag{2.7.9}$$

and tend to limits uniformly for $|x| \to \infty$. Let

$$M = \max\left\{\sup_{-\infty < x < 0} |\phi_1(x) + \alpha_1|; \ \sup_{0 < x < \infty} |\phi_2(x) - \alpha_2|; \ \alpha_1; \ \alpha_2\right\}. \tag{2.7.10}$$

From (2.7.9) and the maximum principle we have the a priori estimate

$$|U_i(x,t)| < M \quad (i = 1, 2) \tag{2.7.11}$$

throughout its domain of definition.

Further, let $u_{i,1}$ and z_1 be defined by the conditions

$$u_{1,1} \equiv 0; \quad u_{1,2} = M_1(x - z_1(t)); \quad z_1(t) = -\delta_1 - \gamma_1 t, \tag{2.7.12}$$

where

$$M_1 \geqq \sup_{0 < x < \infty} |\dot{\phi}_2(x)|; \quad \delta_1 > 0; \quad \gamma_1 = \text{const} > M_1 \tag{2.7.13}$$

and

$$L = M/M_1. \tag{2.7.14}$$

It is then obvious that

$$u_2(L,t) < u_{1,2}(L,t); \quad u_2(x,t_0) < u_{1,2}(x,t_0) \text{ for } 0 < x < L. \tag{2.7.15}$$

In addition, throughout the domain of definition

$$u_{1,1} > u_1. \tag{2.7.15*}$$

For arbitrary given $T > 0$, choose

$$L_1 \geqq \gamma_1 T + \delta_1. \tag{2.7.16}$$

Applying Lemma 1 to the functions u_1, u_2, y and $u_{1,1}$, $u_{2,1}$, z_1 on the interval $-L_1 < x < L$, $t_0 \leqq t \leqq T$, we see that on this interval

$$y(t) > -L_1. \tag{2.7.17}$$

Similarly, we find the following estimate of $y(t)$ from above:

$$y(t) < L_2 \text{ for } t_0 \leqq t \leqq T. \tag{2.7.18}$$

By examining u_1 more closely in the interval $-L_1 < x < y(t)$ and u_2 in the domain $y(t) < x < L_2$ for $t_0 \leqq t \leqq T$, and using estimate (2.7.11) we can carry over the proof of the uniform boundedness of the terms $\partial u_i(x,t)/\partial x$ $(i = 1, 2)$, given for problem B_1. Hence it is possible to define u_i and y on the interval (t_0, T), which completes the proof, since T can be chosen arbitrarily large.

We now study the asymptotic behavior of the solution to problem C for $t \to \infty$.

THEOREM 9 [84].[6] *If*

$$\alpha_2 - a\alpha_1 > 0 \tag{2.7.19}$$

and u_1, u_2, y *are solutions of problem* C, *then*

$$\lim_{t\to\infty} y = -\infty; \quad \lim_{t\to\infty} u_2(x,t) = \frac{\alpha_2 \operatorname{erf}(|\beta_0|/a)}{1 + \operatorname{erf}(|\beta_0|/a)}. \tag{2.7.20}$$

But if

$$\alpha_2 - a\alpha_1 < 0, \tag{2.7.21}$$

then

$$\lim_{t\to\infty} y = +\infty; \quad \lim_{t\to\infty} u_1(x,t) = \frac{\alpha_1 \operatorname{erf}(\beta_0/a)}{1 - \operatorname{erf}(\beta_0/a)}. \tag{2.7.22}$$

Here β_0 *is a root of the equation*

$$F(\beta, \alpha_1, \alpha_2) \equiv \beta - \frac{\alpha_1 \exp(-\beta^2)}{\sqrt{\pi}(1 + \operatorname{erf}\beta)} + \frac{\alpha_2 \exp(-\beta^2/a^2)}{a\sqrt{\pi}(1 - \operatorname{erf}(\beta/a))}. \tag{2.7.23}$$

---

[6] We reproduce the proof of Theorem 9 from [84]. In that paper the case of a general quasilinear equation of parabolic type was examined. For the equation of heat conduction the analysis was given in [42].

For simplicity, we choose $t_0 = 0$. We note that

$$F'_\beta > 0; \quad \beta_0 < 0 \text{ for } \alpha_2 - a\alpha_1 > 0; \quad \beta_0 > 0 \text{ for } \alpha_2 - a\alpha_1 < 0. \tag{2.7.24}$$

Suppose for definiteness that $\alpha_2 - a\alpha_1 > 0$, so that $\beta_0 < 0$. We determine

$$\begin{aligned} v_1 &= v_1(x,t,\beta_0,\alpha_1,\alpha_2); \\ v_2 &= v_2(x,t,\beta_0,\alpha_1,\alpha_2); \end{aligned} \quad z = z(t,\beta_0,\alpha_1,\alpha_2), \tag{2.7.25}$$

as the solution of the problem

$$\begin{aligned} &L_1 v_1 = 0, \ -\infty < x < z; \quad L_2 v_2 = 0, \ z < x < \infty, \ t > 0; \\ &v_1 = -\alpha_1; \quad v_2 = \alpha_2 \text{ for } t = 0; \quad v_i = 0 \text{ for } x = z, \ t > 0; \\ &\dot{z}(t) = \left(\frac{\partial v_1}{\partial x} - \frac{\partial v_2}{\partial x}\right)\Big|_{x=z}, \ t > 0, \ z|_{t=0} = 0. \end{aligned} \tag{2.7.26}$$

It is easily seen that

$$\begin{aligned} v_1 &= \frac{-\alpha_1(\operatorname{erf}\beta_0 - \operatorname{erf}(x/2\sqrt{t}))}{1 + \operatorname{erf}\beta_0}; \\ v_2 &= \frac{\alpha_2(\operatorname{erf}(x/2a\sqrt{t}) - \operatorname{erf}(\beta_0/a))}{1 - \operatorname{erf}(\beta_0/a)}; \end{aligned} \quad z = 2\beta_0\sqrt{t}, \tag{2.7.27}$$

where β_0 is a root of equation (2.7.23). From (2.7.24) it obviously follows that if we replace β_0 by $\beta < \beta_0$ in the definition of v_1, v_2 and z, we then have

$$\dot{z} < \left(\frac{\partial v_1}{\partial x} - \frac{\partial v_2}{\partial x}\right)\Big|_{x=z}. \tag{2.7.27*}$$

If we choose $\epsilon > 0$ arbitrarily small, then there exist numbers $T > 0$ and $N > 0$ so large that

$$\begin{aligned} &u_2(x,t) < \alpha_2 + \epsilon \text{ for } t \geqq T, \quad y < x < \infty; \\ &u_1(x,t) < -\alpha_1 + \epsilon \text{ for } x < -N < y(t), \quad 0 \leqq t \leqq T. \end{aligned} \tag{2.7.28}$$

In fact, let $L > 0$ be so small that

$$\phi_2(x) < \alpha_2 + \epsilon/2 \text{ for } x > L. \tag{2.7.29}$$

Moreover, let

$$\phi_2(x) < \alpha_2 + M; \quad 0 \leqq x < L; \quad \phi_2(L) < \frac{2M + \epsilon}{4} + \alpha_2. \tag{2.7.30}$$

We will examine the function

$$U_2(x,t) = \alpha_2 + M\int_0^L E(x-\xi, a^2t)\,d\xi + \frac{\epsilon}{2}\int_L^\infty E(x-\xi, a^2t)\,d\xi. \tag{2.7.31}$$

We have

$$L_2U_2 = 0; \qquad U_2 > 0 \text{ for } -\infty < x < \infty,\ t > 0;$$

$$U_2|_{t=0} = \alpha_2 + \begin{cases} M \text{ for } 0 < x < L \\ \dfrac{M+\epsilon}{2} \text{ for } x = L; \\ \dfrac{\epsilon}{2} \text{ for } x > L \end{cases} \qquad \text{i.e. } U_2|_{t=0} > \phi_2. \tag{2.7.32}$$

In particular,

$$U_2|_{x=y(t)} > 0 = u_2|_{x=y(t)}. \tag{2.7.33}$$

Comparing (2.7.32) and (2.7.33) with conditions (2.1.5) and (2.7.1) of problem C, we see that throughout its domain of definition

$$u_2(x,t) < U_2(x,t). \tag{2.7.34}$$

On the other hand, from (2.7.31) it follows that

$$U_2 < \alpha_2 + \frac{ML}{2a\sqrt{\pi t}} + \frac{\epsilon}{2}. \tag{2.7.35}$$

Consequently for $ML/2a\sqrt{\pi T} < \epsilon/2$ and $t \geqq T$ we will have

$$U_2 < \alpha_2 + \epsilon. \tag{2.7.36}$$

Let us fix the value of T. Using estimates (2.7.16) and (2.7.17), we choose N_1 so large that

$$y(t) > -N_1 \text{ for } 0 < t \leqq T. \tag{2.7.37}$$

Furthermore, we choose some small $\epsilon_1 > 0$, whose value is defined below, and $N_2 > N_1$ so large that

$$\phi_1(x) < -\alpha_1 + \epsilon_1 < 0 \text{ for } x < -N_2. \tag{2.7.38}$$

We define the function $U_1(x,t)$ by the conditions

$$\begin{gathered} L_1U_1 = 0 \text{ for } -\infty < x < -N_1,\ t > 0; \\ U_1 = 0 \text{ for } x = -N_1,\ t > 0; \\ U_1 = \begin{cases} \alpha - \alpha_1 + \epsilon_1 \text{ for } x < -N_2,\ t = 0 \\ 0 \text{ for } -N_2 < x < -N_1,\ t = 0. \end{cases} \end{gathered} \tag{2.7.39}$$

A uniformly bounded solution of problem (2.7.39) is defined by the equation

$$U_1 = (-\alpha_1 + \epsilon_1) \int_{-\infty}^{-N_2} G(x, \xi, t)\, d\xi, \tag{2.7.40}$$

where

$$G(x, \xi, t) = E(x - \xi, t) - E(x + \xi + 2N_1, t). \tag{2.7.41}$$

Thus for $x < -N_2$ we find that

$$U_1(x,t) = \frac{-\alpha_1 + \epsilon_1}{2}\left[1 + \operatorname{erf}\frac{|x - N_2|}{2\sqrt{t}} - \operatorname{erfc}\frac{|x + 2N_1 - N_2|}{2\sqrt{t}}\right]. \tag{2.7.42}$$

We now choose $N > N_2$ so large that

$$\operatorname{erfc}\frac{|N + N_2 - 2N_1|}{2\sqrt{T}} < \epsilon_1; \qquad \operatorname{erf}\frac{N + N_2}{2\sqrt{T}} > 1 - \epsilon_1. \tag{2.7.43}$$

We then obtain

$$U_1(x, T) < (-\alpha_1 + \epsilon_1)(1 - \epsilon_1). \tag{2.7.44}$$

If we choose ϵ_1 such that the inequality

$$\epsilon_1(1 + \alpha_1) < \epsilon \tag{2.7.45}$$

is satisfied, we find that

$$U_1(x, T) < -\alpha_1 + \epsilon \text{ for } x < -N. \tag{2.7.46}$$

On the other hand, from the condition defining u_1 and from (2.7.37) it follows by the maximum principle that

$$u_1(x,t) < 0 \text{ for } x < -N_1, \quad 0 \leqq t \leqq T. \tag{2.7.47}$$

Comparing the definition (2.7.39) with definitions (2.1.5), (2.7.1) and (2.7.47), we find, again by the maximum principle, that

$$u_1(x,t) < U_1(x,t) \text{ for } 0 < t \leqq T, \quad -\infty < x < -N_1. \tag{2.7.48}$$

Using (2.7.48) and (2.7.46), we see that

$$u_1(x, T) < -\alpha_1 + \epsilon < 0 \text{ for } x < -N, \tag{2.7.49}$$

which completes the proof.

Now let β_1 be a root of the equation

$$F(\beta, \alpha_1 - \epsilon, \alpha_2 + \epsilon) = 0, \tag{2.7.50}$$

where F is defined in agreement with (2.7.33). Let β be such that

$$\beta_1 - \epsilon < \beta < \beta_1 \tag{2.7.51}$$

and define $u_{1,1}$, $u_{2,1}$ and z_1 by the equations

$$\begin{aligned} u_{1,1} &= v_1(x+N, t-T, \beta, \alpha_1 - \epsilon, \alpha_2 + \epsilon); \\ u_{2,1} &= v_2(x+N, t-T, \beta, \alpha_1 - \epsilon, \alpha_2 + \epsilon); \\ z_1 &= 2\beta\sqrt{t-T}, \end{aligned} \tag{2.7.52}$$

where v_1 and v_2 are defined by conditions (2.7.26). Since $\beta < \beta_1$, it follows from (2.7.27*) that

$$\dot{z}_1 < \left(\frac{\partial}{\partial x} u_{1,1} - \frac{\partial}{\partial x} u_{2,1}\right)\Bigg|_{x=z_1}. \tag{2.7.53}$$

Comparing (2.7.53), (2.7.28) and (2.7.26), we see that $u_{1,1}$, $u_{2,1}$ and z_1 satisfy every condition of Lemma 1. Consequently

$$u_1 < u_{1,1}; \quad u_2 < u_{2,1}; \quad y > 2\beta\sqrt{t-T} - N \text{ for } t > T. \tag{2.7.54}$$

It is obvious that the root β_1 of equation (2.7.50) is a continuous function of ϵ, tending to β_0 for $\epsilon \to 0$. Thus (2.7.54) can be written in the form

$$\begin{aligned} u_1(x,t) &< v_1(x+N, t-T, \beta_0, \alpha_1, \alpha_2) - \rho(\epsilon); \\ u_2(x,t) &< v_2(x+N, t-T, \beta_0, \alpha_1, \alpha_2) - \rho(\epsilon); \\ y(t) &> 2(\beta_0 - \rho(\epsilon))\sqrt{t-T} - N, \end{aligned} \tag{2.7.55}$$

where

$$\rho(\epsilon) = O(\epsilon); \qquad \epsilon \to 0. \tag{2.7.56}$$

Now by (2.7.27) we find that

$$u_1(x,t) < \frac{-\alpha_1\left[\operatorname{erf}\dfrac{|x+N|}{2\sqrt{t-T}} - \operatorname{erf}|\beta_0|\right]}{1 - \operatorname{erf}|\beta_0|} \text{ for } x < -N - 2|\beta_0|\sqrt{t-T};$$

$$u_2(x,t) < \frac{\alpha_2\left[\operatorname{erf}\dfrac{|x+N|}{2a\sqrt{t-T}} + \operatorname{erf}\dfrac{|\beta_0|}{a}\right]}{1 + \operatorname{erf}\dfrac{|\beta_0|}{a}} \text{ for } x > -N - 2|\beta_0|\sqrt{t-T}. \tag{2.7.57}$$

Hence we conclude that

$$\overline{\lim_{t\to\infty}} u_2(x,t) \leqq \frac{\alpha_2 \operatorname{erf}(|\beta_0|/a)}{1 + \operatorname{erf}(|\beta_0|/a)}. \tag{2.7.58}$$

In a completely analogous manner we can prove that

$$
\begin{aligned}
&u_1 > v_1(x - N', t - T', \beta_0, \alpha_1, \alpha_2) - \rho'(\epsilon);\\
&u_2 > v_2(x - N', t - T', \beta_0, \alpha_1, \alpha_2) - \rho'(\epsilon);\\
&y(t) < 2(\beta_0 + \rho'(\epsilon)\sqrt{t - T}) + N',
\end{aligned}
\tag{2.7.59}
$$

where $T' > 0$ and $N' > 0$ are sufficiently large and $\rho'(\epsilon) = O(\epsilon)$. Consequently

$$
\varliminf_{t \to \infty} u_2(x, t) \geqq \frac{\alpha_2 \operatorname{erf}(|\beta_0|/a)}{1 + \operatorname{erf}(|\beta_0|/a)} \tag{2.7.60}
$$

as was asserted.

In conclusion we note that from (2.7.55) and (2.7.59) we can easily obtain the asymptotic equality

$$
y(t)/2\beta_0\sqrt{t} \approx 1. \tag{2.7.61}
$$

CHAPTER III

THE INITIAL VELOCITY OF ADVANCE OF THE INTERPHASE BOUNDARY [127]

§1. The integral equations of Problems B_1 and A_1 without the continuous agreement of boundary and initial data

Our immediate goal is to study the initial velocity of advance of the interphase boundary for problems B_1 and A_1 under the assumption of existence of the solution. However, our analysis must be somewhat more careful for these problems than for the integral equations examined above.

We will omit condition (2.2.1) in which we assume continuous agreement of boundary and initial data for the problem. In agreement with Remark 2 on p. 145, the integral equations $(2.2.3_i)$ $(i = 1, 2, 3)$ assume the form[1)]

$$
\begin{aligned}
v_1(t) &= 2[\phi_1(0) - f_1(0)]g_2(y(t),0,t) - 2\phi_1(l)g_2(y(t),l,t) \\
&\quad - 2\int_0^t \dot{f}_1(\tau)\, g_2(y(t),0,t-\tau)\, d\tau \\
&\quad + 2\int_0^l \dot{\phi}_1(\xi)\, g_2(y(t),\xi,t)\, d\xi \\
&\quad - 2\int_0^t v_1(\tau)\frac{\partial}{\partial\xi} g_2(y(t),y(\tau),t-\tau)\, d\tau; \qquad (3.1.1_1)
\end{aligned}
$$

$$
\begin{aligned}
v_2(t) &= 2(f_2(0) - \phi_2(1))\, G_2(y(t),1,t) + 2\phi_2(l)\, G_2(y(t),l,t) \\
&\quad + 2\int_0^t \dot{f}_2(\tau)\, G_2(y(t),1,t-\tau)\, d\tau \\
&\quad + 2\int_l^1 \dot{\phi}_2(\xi)\, G_2(y(t),\xi,t)\, d\xi \\
&\quad + 2a^2\int_0^t v_2(\tau)\frac{\partial}{\partial\xi} G_2(y(t),y(\tau),t-\tau)\, d\tau; \qquad (3.1.1_2)
\end{aligned}
$$

$$
y(t) = l + \int_0^t [v_1(\tau) - v_2(\tau)]\, d\tau. \qquad (3.1.1_3)
$$

[1)] We take $t_0 = 0$.

If $l \neq 0, 1$, then the fact that $\phi_1(0) \neq f_1(0)$ and $\phi_2(1) \neq f_2(0)$ does not affect the process of solution of the system $(3.1.1_i)$ in the small, and equivalently the solution of the original problem. Conversely the inequalities $\phi_i(l) \neq 0$ $(i = 1$ or $2)$ become the most essential factors in determining the nature of these integral equations. In this case the direct proof of the convergence of the Picard iteration process no longer goes through, nor does the proof of equivalence of the solution of the system $(3.1.1_i)$ with the solution of the original problem. Additional difficulties arise in the case of problem A_1 when the initial domain of existence of one of the phases degenerates to a point. In this case the theorems of equivalence, existence and uniqueness require revision.

The problem of existence of the solution is examined in the following chapter. Here our investigation of the initial velocity of the phase boundary is prefaced by a study of the equivalence of the solution of the original boundary problem and the system of integral equations. We prove the following equivalence theorem.

THEOREM 10. *Let v_1, v_2, y be a solution of the system $(3.1.1_i)$ $(i = 1, 2, 3)$, and let u_1 and u_2 be defined by the quadratures $(2.2.4_i)$ $(i = 1, 2)$. Assume that*

$$w_i(t) = \sqrt{t}v_i(t) \quad (i = 1, 2) \tag{3.1.2$_1$}$$

is bounded for $0 \leqq t \leqq T$ and, if $l = 0$, that

$$\zeta_1(t, \tau) = \frac{y(t) - y(\tau)}{2\sqrt{t - \tau}}; \quad \zeta(t, \tau) = \frac{y(t) + y(\tau)}{2\sqrt{t - \tau}} \tag{3.1.2$_2$}$$

is monotonic at the time $t = 0$ for $0 \leqq \tau \leqq t$. Then u_1, u_2, y is a solution to the original boundary problem.

We will not carry out the proof of Theorem 10 in every detail, but limit ourselves to an examination of those parts of the proof which distinguish the problem with discontinuous conditions or with a single initial phase from the problems examined in Chapters I and II. As in Chapter I we prove that u_1, u_2 and y satisfy every condition of the problem except possibly the condition

$$u_1(y(t), t) = u_2(y(t), t) = 0. \tag{3.1.3}$$

If we assume that

$$u_1(y(t), t) = u_1^*(t); \quad u_2(y(t), t) = u_2^*(t) \tag{3.1.3*}$$

and argue as in Chapter I for the proof of Theorem 2, we find that u_1^* can be determined from the homogeneous integral equation

$$u_1^*(t) = 2\int_0^t u_1^*(\tau)\left[\frac{\partial}{\partial\xi}g_1(y(t),y(\tau),t-\tau) - \dot{y}(\tau)\,g_1(y(t),y(\tau),t-\tau)\right]d\tau. \tag{3.1.4}$$

An analgous equation is satisfied by $u_2^*(t)$.

Thus for the proof of Theorem 10 it is sufficient to show that under the given conditions the solution to equation (3.1.4) is unique. With this goal we note first of all that

$$\begin{cases} 2\left[\dfrac{\partial}{\partial\xi}g_1(y(t),y(\tau),t-\tau) - \dot{y}(\tau)\,g_1(y(t),y(\tau),t-\tau)\right] \\ = \dfrac{2}{\sqrt{\pi}}\exp(-\zeta^2)\dfrac{d\zeta}{d\tau} + \dfrac{2}{\sqrt{\pi}}\exp(-\zeta_1^2)\dfrac{d\zeta_1}{d\tau} \\ = K_1(t,\tau) + K_2(t,\tau), \end{cases} \tag{3.1.5}$$

$$\begin{cases} \zeta = \dfrac{y(t)+y(\tau)}{2\sqrt{t-\tau}}; \quad \dfrac{d\zeta}{d\tau} = \left[\dfrac{y(t)-y(\tau)}{2(t-\tau)} + \dot{y}(\tau)\right]\dfrac{1}{2\sqrt{t-\tau}}; \\ \zeta_1 = \dfrac{y(t)-y(\tau)}{2\sqrt{t-\tau}}; \quad \dfrac{d\zeta_1}{d\tau} = \left[\dfrac{y(t)-y(\tau)}{2(t-\tau)} - \dot{y}(\tau)\right]\dfrac{1}{2\sqrt{t-\tau}}. \end{cases} \tag{3.1.6}$$

We examine the two cases: $y(0) = l \neq 0$ and $l = 0$.

First let $l \neq 0$. We set

$$z(t) = \sqrt{t}\,\dot{y}(t); \qquad T_0^* = l^2(16K_0^2)^{-1}; \qquad K_0 = \max|z(t)| \quad \text{for } 0 \leqq t \leqq T_0^*. \tag{3.1.7}$$

For $0 \leqq t \leqq T_0^*$ we have

$$y(t) + y(\tau) > 2[l - 2K_0\sqrt{t}] \geqq l. \tag{3.1.8}$$

From (3.1.5)—(3.1.8) it follows that

$$|K_1(t,\tau)| < \frac{2}{\sqrt{\pi}}\left[\frac{\alpha_2}{\sqrt{t-\tau}} + \frac{\alpha_1 K_0}{\sqrt{\tau}}\right]\frac{1}{l}, \tag{3.1.9}$$

where α_m is defined by the equation

$$\alpha_m = \max_{0<z<\infty} z^m \exp(-z^2). \tag{3.1.10}$$

Consequently

$$\left|\int_0^t u_1^* K_1 d\tau\right| < N\sqrt{t}\max_{0\leqq\tau\leqq T_0^*}|u_1^*(\tau)|, \tag{3.1.11}$$

where

$$N = \frac{4}{l\sqrt{\pi}}(\alpha_2 + \alpha_1 K_0). \tag{3.1.11*}$$

Now let $z(0) \neq 0$. By the theorem of the mean

$$\frac{y(t) - y(\tau)}{2(t-\tau)} = \frac{1}{t-\tau}\int_\tau^t \frac{z(\zeta)\,d\zeta}{2\sqrt{\zeta}} = \frac{z(\tau^*)}{\sqrt{t}+\sqrt{\tau}}, \qquad \tau < \tau^* < t;$$

consequently

$$\left|\frac{y(t) - y(\tau)}{2(t-\tau)} - \frac{z(\tau)}{\sqrt{\tau}}\right| > \left|z(0) - 2\max_{\tau',\tau''\in[0,t]}|z(\tau') - z(\tau'')|\right| \frac{\sqrt{t}}{\sqrt{\tau}(\sqrt{t}+\sqrt{\tau})}. \tag{3.1.12}$$

By virtue of the continuity of $z(t)$ for $t \geqq 0$, which is implied, as is easily seen, by the requirement of boundedness of $w_i(t)$ for $t \geqq 0$, it is possible to find some $T_0^{**} > 0$ so small that for τ', $\tau'' \in (0, T_0^{**})$ we have

$$|z(\tau') - z(\tau'')| < \frac{z(0)}{2}. \tag{3.1.13}$$

But in this case $d\zeta_1/d\tau$ (see (3.1.6)) will be of constant sign, while at the same time

$$\left|\int_0^t u_1^* K_2 d\tau\right| \leqq \max_{0\leqq\tau\leqq T_0^{**}}|u_1^*(\tau)|\,\mathrm{erf}\frac{|y(t) - l|}{2\sqrt{t}}. \tag{3.1.14}$$

Now let $z(0) = 0$. In this case, by the theorem of the mean

$$\left|\frac{y(t) - y(\tau)}{2(t-\tau)} - \frac{z(\tau)}{\sqrt{\tau}}\right| = \left|\frac{z(\tau^*)}{\sqrt{t}+\sqrt{\tau}} - \frac{z(\tau)}{\sqrt{\tau}}\right| < \frac{2\max|z(\tau)|}{\sqrt{\tau}}. \tag{3.1.15}$$

Using (3.1.5) and (3.1.6), we therefore find that

$$\left|\int_0^t u_1^* K_2 d\tau\right| < \frac{2}{\sqrt{\pi}}\max_{0\leqq\tau\leqq T_0^{**}}|u_1^*| \max|z(\tau)| \int_0^t \frac{d\tau}{\sqrt{\tau}\sqrt{t-\tau}} = 2\sqrt{\pi}\max_{0\leqq\tau\leqq T_0^{**}}|u_1^*| \max|z(\tau)|. \tag{3.1.16}$$

Combining estimates (3.1.11), (3.1.14) and (3.1.16) and setting

$$T_1^* = \min(T_0^*, T_0^{**}),$$

we obtain

$$|u_1^*(t)| < \max_{0 \leqq \tau \leqq T_1^*} |u_1^*(\tau)| (N\sqrt{t} + \psi), \tag{3.1.17}$$

where

$$\psi = \begin{cases} \operatorname{erf} \dfrac{|y(t) - l|}{2\sqrt{t}} & \text{for } z(0) \neq 0; \\ 2\sqrt{\pi} \max\limits_{0 \leqq \tau \leqq T_1^*} |z(\tau)| & \text{for } z(0) = 0. \end{cases} \tag{3.1.17*}$$

But

$$\begin{aligned} &\lim_{t \to 0} \frac{|y(t) - l|}{2\sqrt{t}} = \operatorname{erf}|z(0)| < 1 \quad \text{for } z(0) \neq 0. \\ &\lim_{t \to 0} z(t) = 0 \qquad\qquad\qquad\quad \text{for } z(0) = 0. \end{aligned} \tag{3.1.18}$$

Consequently it is possible to find some $T_0 < T_1^*$ such that for $0 \leqq t \leqq T_0$ we have

$$\max_{0 \leqq \tau \leqq T_0} |u_1^*(\tau)| < \max_{0 \leqq \tau \leqq T_0} |u_1^*(\tau)|, \tag{3.1.19}$$

provided that $\max|u_1^*| > 0$. But this shows that

$$u_1^*(t) \equiv 0 \quad \text{for } l \neq 0. \tag{3.1.20}$$

Now let $l = 0$. By virtue of the assumption of $(3.1.2_2)$ the terms $d\zeta/d\tau$ and $d\zeta_1/d\tau$ do not change sign at the moment $t = 0$. Consequently it is possible to find some $T > 0$ such that for $0 \leqq t \leqq T$ we will have

$$\begin{aligned} \left| \int_0^t u_1^* K_1 d\tau \right| &\leqq \max_{0 \leqq \tau \leqq T} |u_1^*| \frac{2}{\sqrt{\pi}} \int_0^t \exp(-\zeta^2) \frac{d\zeta}{d\tau} d\tau \\ &= \max_{0 \leqq \tau \leqq T} |u_1^*| \operatorname{erfc} \frac{y(t)}{2\sqrt{t}}; \\ \left| \int_0^t u_2^* K_2 d\tau \right| &\leqq \max_{0 \leqq \tau \leqq T} |u_1^*| \frac{2}{\sqrt{\pi}} \int_0^t \exp(-\zeta_1^2) \frac{d\zeta_1}{d\tau} d\tau \\ &= \max_{0 \leqq \tau \leqq T} |u_1^*| \operatorname{erf} \frac{y(t)}{2\sqrt{t}}. \end{aligned} \tag{3.1.21}$$

We note that the equality sign is only possible for $u_1^* \equiv \text{const}$, for otherwise the point set on which $|u_1^*(\tau)| < \max|u_1^*(\tau)|$ will have positive measure. Thus for $l = 0$ and $u_1^* \not\equiv \text{const}$ we have

$$\left| \int_0^t u_1^*(K_1 + K_2)\, d\tau \right| < \max_{0 \le \tau \le T} |u_1^*| \left(\operatorname{erfc} \frac{y(t)}{2\sqrt{t}} + \operatorname{erf} \frac{y(t)}{2\sqrt{t}} \right) \equiv \max_{0 \le \tau \le T} |u_1^*|. \tag{3.1.22}$$

Inserting this estimate into (3.1.4), we find that

$$0 < \max|u_1^*(\tau)| < \max|u_1^*(\tau)|,$$

which is impossible. Thus, if $l = 0$,

$$u_1^*(t) \equiv \text{const}. \tag{3.1.23}$$

But in this case it follows from (3.1.4) that

$$u_1^* \equiv u_1^* \left(1 - 2 \operatorname{erf} \frac{y(t)}{2\sqrt{t}} \right), \tag{3.1.24}$$

which is possible only for $u_1^* \equiv 0$; hence the proof is complete.

REMARK. In what follows we investigate the case

$$y(0) = 0, \quad \dot{y}(t) = \gamma t^{\nu}(1 + y_1(t)), \quad \nu \geqq -\frac{1}{2}, \quad \lim_{t \to 0} y_1(t) = 0.$$

It is obvious that in this case condition $(3.1.2_2)$ is realized, and consequently the equivalence theorem holds.

§2. Initial speed of the interphase boundary. Formulation of the result

The formulation and proof of an existence and uniqueness theorem in the singular case in question is conveniently prefaced by a study of the initial velocity of propagation of the interphase boundary, under the assumption of existence of the solution to the problem.

The initial velocity of motion of the boundary $x = y(t)$ in the problems B_1, B_2 and C is known if the condition of agreement

$$\phi_i(l) = 0 \qquad (i = 1, 2) \tag{3.2.1}$$

is valid. In this case

$$\dot{y}(0) = \dot{\phi}_1(l) - \dot{\phi}_2(l)$$

and if $\gamma = \dot{y}(0) \neq 0$ then

$$(3.2.1^*) \qquad y(t) = l + \gamma t(1 + y^*(t)); \qquad \lim_{t \to 0} y^*(t) = 0.$$

But for $\gamma = 0$ the order of contact of the curves $x = l$ and $x = y(t)$ for $t = 0$ becomes unknown. In addition, it is unknown when condition (3.2.1) is omitted, or in the case of problem A_1 or A_3 when the domain of existence of one of the phases degenerates to a point. Before we carry out a detailed investigation of the initial velocity of propagation of the interphase boundary in problems A_1 and A_3 we will state the main results for problems B_i and C.[2)]

Suppose in the case of problem A_1 that at the moment $t = 0$

$$(3.2.2_1) \qquad f_1(t) = -\psi_1(t)\, t^{m_0}; \qquad \phi(x) = \psi_2(x) \cdot x^{n_0},$$

where

$$(3.2.2_2) \qquad m_0 > -1; \qquad n_0 > -1,$$

and for any $\epsilon > 0$

$$(3.2.2_3) \qquad \lim_{t \to 0} \psi_i(t) \cdot t^{\epsilon} = 0; \qquad \lim_{t \to 0} \psi_i(t)\, t^{-\epsilon} = \infty \quad (i = 1, 2).$$

Assume further that there exists a solution to problems A_1 and A_3 such that

$$(3.2.2_4) \qquad z(t) = \sqrt{t}\dot{y}(t)$$

is continuous for $t \geqq 0$. Under this assumption we have

THEOREM 11. *Let $x = y(t)$ be the boundary in problem A_1. Then the following assertions are valid.*

a) *If*

$$(3.2.3_1) \qquad f_1(0) \neq 0,$$

then

$$(3.2.3_2) \qquad \dot{y}(t) = \gamma t^{-1/2}(1 + y_1^*(t)); \qquad \lim_{t \to 0} y_1^* = 0,$$

for γ a root of the equation

[2)] The interest in problems B_i and C is only for the case when $\phi_i(l) \neq 0$. In fact, for $\phi_1(l) = \phi_2(l) = 0$ the existence and uniqueness is proved independently of the investigation of the initial velocity of propagation of the interphase boundary.

$$(3.2.3_3) \qquad \gamma = -\frac{f_1(0)}{\sqrt{\pi}}\frac{\exp(-\gamma^2)}{\operatorname{erf}\gamma} - \frac{\phi(0)}{a\sqrt{\pi}}\frac{\exp\left(-\frac{\gamma^2}{a^2}\right)}{\operatorname{erfc}\frac{\gamma}{a}}$$

b) *If*

$$(3.2.4_1) \qquad f_1(0) = 0; \qquad \left|\frac{d\ln f_1}{dt}\right| < Kt^{-1} \quad (K = \text{const} > 0),$$

then

$$(3.2.4_2) \qquad \dot{y}(t) = \gamma(t)\, t^{\nu_0},$$

where for arbitrary $\epsilon > 0$

$$(3.2.4_3) \qquad \lim_{t\to 0}\gamma(t)\, t^{\epsilon} = 0; \qquad \lim_{t\to 0}\gamma(t)\, t^{-\epsilon} = \infty,$$

and

$$(3.2.4_4) \qquad \nu_0 = \begin{cases} m_0 + 1/2 & \text{for } \phi(0) \neq 0 \\ m_0 - n_0/2 & \text{for } \phi(0) = 0; \quad n_0 \leqq m_0 \\ m_0/2 & \text{for } \phi(0) = 0; \quad n_0 < m_0. \end{cases}$$

c) *If in* (3.2.2$_1$) *the limits*

$$(3.2.5_1) \qquad \lim_{t\to 0}\psi_1(t) = c_0 > 0; \qquad \lim_{x\to 0}\psi_2(x) = b_0 > 0$$

exist, and if (3.2.4$_1$) *holds, then*

$$(3.2.5_2) \qquad \dot{y}(t) = \gamma t^{\nu_0}(1 + y_1^*(t)); \quad \lim_{t\to 0} y_1^* = 0,$$

for ν_0 *defined in agreement with* (3.2.4$_4$) *and*

$$(3.2.5_3) \qquad \gamma = \begin{cases} \sqrt{\dfrac{c_0(m_0+2)}{2(m_0+1)}} & \text{for } \phi(0) = 0; \quad m_0 < n_0 \\ -\dfrac{\alpha_2}{2} + \sqrt{\dfrac{\alpha_2^2}{4} + \dfrac{c_0(m_0+2)}{2(m_0+1)}} & \text{for } \phi(0) = 0; \quad m_0 = n_0 \\ \dfrac{c_0}{2\alpha_2}\cdot\dfrac{2(m_0+1) - n_0}{m_0+1} & \text{for } \phi(0) = 0; \quad m_0 > n_0 \\ \dfrac{c_0}{2\alpha_2}\cdot\dfrac{2m_0+3}{m_0+1} & \text{for } \phi(0) \neq 0. \end{cases}$$

Here

$$\alpha_2 = (2a)^{n_0}\Gamma\left(\frac{n_0+1}{2}\right)\frac{b_0}{\sqrt{\pi}} \quad \text{for} \quad \phi(0) = 0;$$

$$\alpha_2 = \frac{\phi(0)}{a\sqrt{\pi}} \quad \text{for} \quad \phi(0) \neq 0. \tag{3.2.5$_4$}$$

REMARK. Let $x = y^*(t)$ be the interphase boundary for problem A_1 on the halfline $x > 0$ with boundary and initial conditions

$$f_1(t) \equiv f_1(0); \qquad \phi(x) \equiv \phi(0); \qquad f_1(0) \neq 0.$$

Then $y^*(t) = 2\gamma\sqrt{t}$, where γ is defined in agreement with (3.2.3$_3$). Thus the curves $x = y(t)$ and $x = y^*(t)$ are in contact for $t = 0$, as was obvious a priori.[3)]

THEOREM 12. *In the conditions of problem* A_3 *let* ϕ *and* f_1 *satisfy the conditions*

$$-f_1(t) = c_0 t^{m_0}(1 + f_1^*(t)); \quad \lim_{t\to 0} f_1^* = 0; \quad m_0 \geqq 0; \quad c_0 > 0;$$

$$\dot{\phi}(x) = b_0 x^{n_0}(1 + \phi^*(x)); \quad \lim_{x\to 0} \phi^* = 0; \quad n_0 > -1; \quad b_0 > 0. \tag{3.2.6}$$

Then the following assertions are valid.

a) *If*

$$\phi(0) = 0; \qquad m_0 \leqq n_0/2, \tag{3.2.6$'_1$}$$

where

$$hc_0 > (2a)^{n_0} b_0 \Gamma\left(\frac{n_0+1}{2}\right)\frac{1}{\sqrt{\pi}} \quad \text{if} \quad m_0 = \frac{n_0}{2}, \tag{3.2.6$''_1$}$$

then

$$\dot{y}(t) = \gamma t^{m_0}(1 + y_1^*(t)); \qquad \lim_{t\to 0} y_1^* = 0, \tag{3.2.6$_2$}$$

where

$$\gamma = \begin{cases} hc_0 & \text{for} \quad m_0 < n_0/2 \\ hc_0 - (2a)^{n_0} b_0 \Gamma\left(\frac{n_0+1}{2}\right)\frac{1}{\sqrt{\pi}} & \text{for} \quad m_0 = n_0/2. \end{cases} \tag{3.2.6$_3$}$$

[3)] This result was first obtained by A. Huber [61] in the same way as assertion (3.2.5$_2$) is obtained in the case when $m_0 = n_0 = 0$.

b) *If condition* $(3.2.6_1')$ *is not realized, or if*

$$(3.2.6_4) \qquad hc_0 > (2a)^{n_0} b_0 \Gamma \left(\frac{n_0+1}{2}\right) \frac{1}{\sqrt{\pi}} \quad \text{for} \quad m_0 = \frac{n_0}{2},$$

then the boundary condition

$$(3.2.7_1) \qquad \left(\frac{\partial}{\partial x} - h\right) u_1|_{x=0} = -hf_1(t); \quad u_2|_{t=0} = \phi(x); \quad y(0) = 0$$

is inconsistent with the Stefan condition

$$(3.2.7_2) \qquad \dot{y}(t) = \left(\frac{\partial}{\partial x} u_1 - \frac{\partial}{\partial x} u_2\right) \Bigg|_{x=y(t)} \equiv v_1(t) - v_2(t); \quad t > 0$$

and it follows that

$$(3.2.7_3) \qquad 0 < y(t) < 1 \quad \text{for} \quad t > 0.$$

c) *If in* $(3.2.6_1'')$ *the inequality sign is replaced by an equality sign, then for the solution of the problem of compatibility of the conditions* $(3.2.7_1)$—$(3.2.7_3)$ *the investigation must be extended.*

THEOREM 13. *Let* $x = y(t)$ *be the interphase boundary in problems* B_1, B_3 *and* C. *Further, let*

$$(3.2.8_1) \qquad \phi_1^2(l) + \phi_2^2(l) \neq 0.$$

Then

$$(3.2.8_2) \qquad \dot{y}(t) = \gamma t^{-1/2}(1 + y_1^*(t)); \quad \lim_{t \to 0} y_1^* = 0,$$

where γ *is a root of the equation*

$$(3.2.8_3) \qquad \gamma = \frac{\phi_1(l) \exp(-\gamma^2)}{\sqrt{\pi}(1 + \operatorname{erf}\gamma)} + \frac{\phi_2(l) \exp\left(-\dfrac{\gamma^2}{a^2}\right)}{a\sqrt{\pi}\left(1 - \operatorname{erf}\dfrac{\gamma}{a}\right)}.$$

§3. Proof of Theorem 11. The case $f_1(0) \neq 0$

Let u_1, u_2, y be a solution of problem A_1 such that $z = \sqrt{t}\dot{y}$ is continuous for $t \geqq 0$. Furthermore, let

$$(3.3.1) \qquad w_i(t) = \sqrt{t} v_i(t) \qquad (i = 1, 2).$$

We will prove that $w_i(t)$ is also continuous for $t \geqq 0$. Obviously it is sufficient to prove the continuity of $w_2(t)$, since the continuity of $w(t)$

follows from that of $z(t)$ and $w_2(t)$ and from the Stefan condition

$$z(t) = w_1(t) - w_2(t). \tag{3.3.2}$$

We note that from (3.3.1) and (3.1.1$_2$) it follows that

$$w_2(t) = F_2(t) + W(t|w_2), \tag{3.3.3}$$

where

$$F_2(t) = 2\sqrt{t}\left\{[f_2(0) - \phi(1)]\,G_2(y(t),1,t) + \phi(0)\,G_2(y(t),0,t) + \int_0^t \dot{f}_2(\tau)\,G_2(y(t),1,t-\tau)\,d\tau + \int_0^1 \dot{\phi}(\xi)\,G(y(t),\xi,t)\,d\xi\right\} \tag{3.3.4$_1$}$$

and

$$W(t|w_2) = 2a^2\int_0^t w_2(\tau)\sqrt{\frac{t}{\tau}}\frac{\partial}{\partial\xi}G_2(y(t),y(\tau),t-\tau)\,d\tau. \tag{3.3.4$_2$}$$

Since $F_2(t)$ is obtained from a uniformly convergent integral, it it continuous for $t \geqq 0$. Furthermore, it is easily seen that if $w_2(\tau)$ is bounded, then $W(t|w_2)$ is continuous for $t \geqq 0$. Consequently for the proof of continuity of $w_2(t)$ it is sufficient to prove the existence and uniqueness of a bounded solution to the integral equation (3.3.3) for arbitrary continuous $F(t)$. It is easily seen that this holds, since for $0 \leqq t \leqq T$, with $T > 0$ sufficiently small, the operator W is a contraction mapping of the space of C-functions, continuous for $0 \leqq t \leqq T$, into itself.

In fact, let

$$|z(t)| \leqq K_0 \quad \text{for} \quad 0 \leqq t \leqq t_0. \tag{3.3.5$_1$}$$

Set

$$\delta w = w' - w \quad \text{for} \quad w,\ w' \in C;$$
$$\|\delta w\| = \max_{0\leqq t\leqq T}|\delta w|. \tag{3.3.5$_2$}$$

We then have

$$\delta W = J_1 + J_2, \tag{3.3.6}$$

$$J_1 = \frac{1}{2a\sqrt{\pi}}\int_0^t \delta w\frac{y(t)-y(\tau)}{(t-\tau)^{3/2}}\exp\left(-\frac{(y(t)-y(\tau))^2}{4a^2(t-\tau)}\right)\sqrt{\frac{t}{\tau}}d\tau;$$
$$J_2 = \frac{1}{2a\sqrt{\pi}}\int_0^t \delta w\frac{2-y(t)-y(\tau)}{(t-\tau)^{3/2}}\exp\left(-\frac{(2-y(t)-y(\tau))^2}{4a^2(t-\tau)}\right)\sqrt{\frac{t}{\tau}}\,d\tau. \tag{3.3.7}$$

We first examine J_1. Using the continuity of z, we choose $T_1 \leqq t_0$ so small that for arbitrary t' and t'' from $[0, T_1]$,

$$|z(t') - z(t'')| < \epsilon, \tag{3.3.8}$$

where $\epsilon > 0$ is a preassigned number which will be chosen later on. Furthermore, let

$$\mu(t, \tau) = \frac{y(t) - y(\tau)}{2(\sqrt{t} - \sqrt{\tau})}. \tag{3.3.9}$$

By the theorem of the mean

$$\mu(t, \tau) = \frac{1}{2(\sqrt{t} - \sqrt{\tau})} \int_\tau^t \frac{z(\zeta)\, d\zeta}{\sqrt{\zeta}} = z(\tau + \Theta(t - \tau)); \quad 0 < \Theta < 1. \tag{3.3.9*}$$

Consequently, for $0 \leqq \tau \leqq t \leqq T_1$

$$|\mu(t, \tau) - \mu(t, 0)| < \epsilon; \qquad |\mu(t, \tau)| < K_0. \tag{3.3.10}$$

We have

$$J_1 = J_{1,0} + J_{1,1} + J_{1,2}, \tag{3.3.11}$$

where

$$J_{1,0} = \frac{1}{a\sqrt{\pi}} \int_0^t \delta w(\tau) \frac{\mu(t, 0)}{\sqrt{t - \tau}} \exp\left(- \frac{\mu^2(t, 0)}{a^2} \frac{\sqrt{t} - \sqrt{\tau}}{\sqrt{t} + \sqrt{\tau}} \right) \sqrt{\frac{t}{\tau}} \frac{d\tau}{\sqrt{t} + \sqrt{\tau}};$$

$$J_{1,1} = \frac{1}{a\sqrt{\pi}} \int_0^t \delta w(\tau) \frac{\mu(t, \tau) - \mu(t, 0)}{\sqrt{t - \tau}} \exp\left(- \frac{\mu^2(t, 0)}{a^2} \frac{\sqrt{t} - \sqrt{\tau}}{\sqrt{t} + \sqrt{\tau}} \right)$$

$$\cdot \sqrt{\frac{t}{\tau}} \frac{d\tau}{\sqrt{t} + \sqrt{\tau}}; \tag{3.3.12}$$

$$J_{1,2} = \frac{1}{a\sqrt{\pi}} \int_0^t \delta w(\tau) \frac{\mu(t, \tau)}{\sqrt{t - \tau}} \left[\exp\left(- \frac{\mu^2(t, \tau)}{a^2} \frac{\sqrt{t} - \sqrt{\tau}}{\sqrt{t} + \sqrt{\tau}} \right) \right.$$

$$\left. - \exp\left(- \frac{\mu^2(t, 0)}{a^2} \frac{\sqrt{t} - \sqrt{\tau}}{\sqrt{t} + \sqrt{\tau}} \right) \right] \sqrt{\frac{t}{\tau}} \frac{d\tau}{\sqrt{t} + \sqrt{\tau}}.$$

Using (3.3.10), we easily find that

$$|J_{1,0}| < \max_{0 \leqq t \leqq T_1} |\delta w| \operatorname{erf} \frac{K_0}{a}; \qquad |J_{1,1}| < \frac{\epsilon\sqrt{\pi}}{a} \max_{0 \leqq t \leqq T_1} |\delta w|;$$

$$|J_{1,2}| < \frac{4 K_0^2 \epsilon}{3a^3\sqrt{\pi}} \max_{0 \leqq t \leqq T_1} |\delta w|. \tag{3.3.13}$$

We now subject $\epsilon > 0$ to the condition

$$0 < \frac{\sqrt{\pi}}{a}\left(1 + \frac{4}{3}\frac{K_0^2}{a^2\pi}\right)\epsilon < \frac{1}{2}\left(1 - \operatorname{erf}\frac{K_0}{a}\right). \tag{3.3.14}$$

Inserting (3.3.13) into (3.3.11) gives us the estimate

$$|J_1| < \frac{1}{2}\left(1 + \operatorname{erf}\frac{K_0}{a}\right) \max_{0<t<T_1} |\delta w|. \tag{3.3.15}$$

We next examine J_2. Let

$$0 < T_2 < \min\left\{t_0, \frac{(1-\beta)^2}{4K_0^2}\right\} \tag{3.3.16}$$

where $0 < \beta < 1$ is an arbitrary constant. By virtue of $(3.3.5_1)$ we have

$$2 - y(t) - y(\tau) \geqq 2\beta > 0 \quad \text{for} \quad 0 \leqq \tau \leqq t \leqq T_2. \tag{3.3.17}$$

This reduces to the estimate

$$|J_2| < \frac{a\alpha_2}{\beta}\sqrt{\pi t} \max_{0 \leqq t \leqq T_2} |\delta w| = M\sqrt{t} \max_{0 \leqq t \leqq T_2} |\delta w|. \tag{3.3.18}$$

Here

$$\alpha_m = \max_{0 \leqq z < \infty} z^m \exp(-z^2); \qquad m = 0, 1, \cdots. \tag{3.3.18*}$$

Now let $0 \leqq t \leqq T \leqq \min(T_1, T_2)$ and

$$M\sqrt{T} + \frac{1}{2}\left(1 + \operatorname{erf}\frac{K_0}{a}\right) \leqq q < 1. \tag{3.3.19}$$

In this case from (3.3.6) and estimates (3.3.15), (3.3.18) and (3.3.19) it follows that

$$\|W\| \leqq q\|w\|, \tag{3.3.20}$$

which proves our assertion.

From the proof of the continuity of $w_i(t)$ under the assumption of continuity of $z(t)$ for $t \geqq 0$, it follows that $v_i(t)$ and $\dot{y}(t)$ can be represented in the form

$$v_i(t) = \frac{\alpha_i + v_i^*(t)}{\sqrt{t}} \quad (i = 1, 2); \qquad \dot{y}(t) = \frac{\gamma + y_i^*(t)}{\sqrt{t}}, \tag{3.3.21}$$

where

$$\lim_{t\to 0} v_i^*(t) = 0; \qquad \lim_{t\to 0} y_i^*(t) = 0. \tag{3.3.21*}$$

We next show that

$$\alpha_1 = -\frac{f_1(0)}{\sqrt{\pi}} \frac{\exp(-\gamma^2)}{\operatorname{erf}\gamma}; \quad \alpha^2 = \frac{\phi(0)}{a\sqrt{\pi}} \frac{\exp\left(-\frac{\gamma^2}{a^2}\right)}{\operatorname{erfc}\frac{\gamma}{a}}. \tag{3.3.22}$$

In fact, for $l = 0$ equations $(3.1.1_i)$ $(i = 1, 2)$ assume the form

$$v_1(t) = \sum_{j=0}^{3} I_j(t); \qquad v_2(t) = \sum_{j=0}^{3} J_j(t), \tag{3.3.23}$$

where

$$I_0(t) = -\frac{2f_1(0)}{\sqrt{\pi t}} \exp\left(-\frac{y^2(t)}{4t}\right);$$

$$I_1(t) = -\frac{2}{\sqrt{\pi}} \int_0^t f_1(\tau) \frac{\exp\left(-\frac{y^2(t)}{4(t-\tau)}\right)}{\sqrt{t-\tau}} d\tau;$$

$$I_2(t|v_1) = \frac{1}{2\sqrt{\pi}} \int_0^t v_1(\tau) \frac{y(t)+y(\tau)}{(t-\tau)^{3/2}} \exp\left(-\frac{(y(t)+y(\tau))^2}{4(t-\tau)}\right) d\tau; \tag{3.3.24$_1$}$$

$$I_3(t|v_1) = -\frac{1}{2\sqrt{\pi}} \int_0^t v_1(\tau) \frac{y(t)-y(\tau)}{(t-\tau)^{3/2}} \exp\left(-\frac{(y(t)-y(\tau))^2}{4(t-\tau)}\right) d\tau;$$

$$J_0(t) = \frac{\phi(0)}{a\sqrt{\pi t}} \exp\left(-\frac{y^2(t)}{4a^2t}\right);$$

$$J_1(t) = \frac{1}{a\sqrt{\pi t}} \int_0^1 \phi(\xi) \exp\left(-\frac{(y(t)-\xi)^2}{4a^2t}\right) d\xi; \tag{3.3.24$_2$}$$

$$J_2(t) = \frac{1}{2a\sqrt{\pi}} \int_0^t v_2(\tau) \frac{y(t)-y(\tau)}{(t-\tau)^{3/2}} \exp\left(-\frac{(y(t)-y(\tau))^2}{4a^2(t-\tau)}\right) d\tau$$

and $J_3(t)$ represents the sum of every other term in the right side of $(3.1.1_2)$. Obviously, for arbitrary p

$$\lim_{t\to 0} J_3(t)\, t^{-p} = 0. \tag{3.3.25}$$

We now choose $\epsilon > 0$ arbitrarily small and $T > 0$ such that

$$|y_1^*| < \epsilon, \quad |v_i^*| < \epsilon \text{ for } 0 \leqq t \leqq T \quad (i = 1, 2). \tag{3.3.26}$$

Introducing (3.3.21) into the expression $I_2(t|v_1)$, we find that

$$I_2 = \frac{(\alpha_1 - \epsilon)(\gamma - \epsilon)}{\sqrt{\pi}} \int_0^t \frac{\exp\left(-(\gamma+\epsilon)^2 \frac{\sqrt{t}+\sqrt{\tau}}{\sqrt{t}-\sqrt{\tau}}\right)}{(\sqrt{t}-\sqrt{\tau})\sqrt{t-\tau}} \frac{d\tau}{\sqrt{\tau}} < I_2$$

$$< \frac{(\alpha_1 + \epsilon)(\gamma + \epsilon)}{\sqrt{\pi}} \int_0 \frac{\exp\left(-(\gamma-\epsilon)^2 \frac{\sqrt{t}+\sqrt{\tau}}{\sqrt{t}-\sqrt{\tau}}\right)}{(\sqrt{t}-\sqrt{\tau})\sqrt{t-\tau}} \frac{d\tau}{\sqrt{\tau}} = \bar{I}_2. \tag{3.3.27}$$

Substituting

$$(\gamma+\epsilon)\sqrt{\frac{\sqrt{t}+\sqrt{\tau}}{\sqrt{t}-\sqrt{\tau}}} = \zeta; \quad (\gamma-\epsilon)\sqrt{\frac{\sqrt{t}-\sqrt{\tau}}{\sqrt{t}-\sqrt{\tau}}} = \zeta_1$$

gives

$$\sqrt{t}I_2 = (\alpha_1 - \epsilon)\frac{\gamma-\epsilon}{\gamma+\epsilon}\operatorname{erfc}(\tau+\epsilon) < \sqrt{t}I_2$$

$$< (\alpha_1 + \epsilon)\frac{\gamma+\epsilon}{\gamma-\epsilon}\operatorname{erfc}(\gamma-\epsilon) = \sqrt{t}I_2.$$

Passing to the limit for $t \to 0$ and noting that ϵ is arbitrarily small, we find that

$$\lim_{t\to 0}\sqrt{t}I_2 = \alpha_1 \operatorname{erfc}\gamma. \tag{3.3.29}$$

Similarly,

$$\lim_{t\to 0}\sqrt{t}I_3 = -\alpha_1 \operatorname{erf}\gamma. \tag{3.3.30}$$

By virtue of $(3.2.2_1)$ and $(3.2.2_2)$ we obviously have

$$\lim_{t\to 0}\sqrt{t}I_1 = 0. \tag{3.3.31}$$

Finally,

$$\lim_{t\to 0}\sqrt{t}I_0 = -\frac{2f_1(0)}{\sqrt{\pi}}\exp(-\gamma^2). \tag{3.3.32}$$

Substituting (3.3.29)—(3.3.32) into (3.3.23) and taking (3.3.21) into account, we find that

$$\alpha_1 = -\frac{2f_1(0)}{\sqrt{\pi}}\exp(-\gamma^2) + \alpha_1(1 - 2\operatorname{erf}\gamma), \tag{3.3.33}$$

which for $f_1(0) \neq 0$ is equivalent to the first of the equations (3.3.22). In exactly the same way we can prove the second of these equations. By substituting them into the Stefan condition, and adding the result to $(3.2.3_2)$, we complete the proof of Theorem 11 in the case $f_1(0) \neq 0$.

§4. Proof of Theorem 11. Case $f_1(0) = 0$

Now let $f_1(0) = 0$. We divide the set of all real λ into classes A_i and B_i, assigning to A_i every λ for which

$$\lim_{t\to 0} v_i(t)\, t^{-\lambda} = 0 \qquad (i = 1, 2). \tag{3.4.1}$$

We claim that the division (A_i, B_i) is a cut. In fact, if

$$\lambda \in A_i; \quad \mu \in B_i; \quad \lambda^* < \lambda; \quad \mu^* > \mu, \tag{3.4.2}$$

then

$$\lambda^* \in A_i; \quad \mu^* \in B_i. \tag{3.4.2$_2$}$$

Thus it suffices to prove the nonemptiness of the classes A_i and B_i. The class A_i is nonempty, since from the continuity of $\sqrt{t}v_i(t)$ for $t \geqq 0$ it follows that for arbitrary $\epsilon > 0$

$$\lim_{t\to 0} v\ t^{1/2+\epsilon} = 0,$$

i.e. that

$$-\left(\frac{1}{2} + \epsilon\right) \in A_i \quad (i = 1, 2). \tag{3.4.3}$$

Furthermore, it is easy to see that the nonemptiness of B_2 implies the nonemptiness of B_1. In fact, let us assume the opposite. Then there exists a λ such that

$$\lim_{t\to 0} v_1(t)\ t^{\lambda} = 0; \quad \lim_{t\to 0} v_2(t)\, t^{\lambda} = \infty. \tag{3.4.4}$$

Introducing these limit relations into the Stefan condition, we find that

$$\lim_{t\to 0} t^{\lambda}\dot{y}(t) = -\lim_{t\to 0} v_2(t)\, t^{\lambda} = -\infty. \tag{3.4.5}$$

But this shows that $\dot{y}(t) < 0$, for every $t > 0$ sufficiently small, which implies that $y(0) = 0$ and $0 < y(t) < 1$ for $t > 0$.

Thus it is now sufficient to prove the nonemptiness of B_2. Assume the opposite. Then for arbitrary $p > -\frac{1}{2}$

$$\lim_{t\to 0} t^{-p} v_2(t) = 0. \tag{3.4.6}$$

Hence if J_2 is defined by $(3.3.4_2)$, then

$$\lim_{t\to 0} t^{-p} J_2(t) = 0 \tag{3.4.7}$$

and together with this, by virtue of (3.3.23) and (3.3.25),

$$\lim_{t\to 0} t^{-p} v_2(t) = \lim_{t\to 0} t^{-p}(J_0(t) + J_1(t)). \tag{3.4.8}$$

If $\phi(0) \neq 0$, then

$$\lim_{t\to 0} t^{1/2}(J_0(t) + J_1(t)) = \frac{\phi(0)}{a\sqrt{\pi}} \exp\left(-\frac{\gamma^2}{a^2}\right), \tag{3.4.9}$$

where as before

$$\gamma = \lim_{t\to 0} \sqrt{t}\dot{y}(t). \tag{3.4.10}$$

By virtue of (3.4.8) this verifies (3.4.6). Thus for $\phi(0) \neq 0$ the class B_i $(i = 1, 2)$ is nonempty.

Now let $\phi(0) = 0$. In this case (3.4.8) is transformed into

$$\lim_{t\to 0} t^{-p} v_2(t) = \lim_{t\to 0} t^{-p} J_1(t). \tag{3.4.11}$$

From the definition $(3.2.2_1)$ it follows that for every $\epsilon > 0$ there exists a $\delta > 0$, $b_1 > 0$ and $b_2 > 0$ such that

$$b_1 x^{n_0+\epsilon} < \dot{\phi}_1(x) < b_2 x^{n_0-\epsilon}; \quad 0 < x < \delta. \tag{3.4.12}$$

We have

$$J_1 = J_1' + J_1'' = \frac{1}{a\sqrt{\pi t}}\left(\int_0^{\delta} + \int_{\delta}^{1} \dot{\phi}(\xi) \exp\left(-\frac{(y(t) - \xi)^2}{4a^2 t}\right) d\xi\right). \tag{3.4.13}$$

It is obvious that for arbitrary p

$$\lim_{t\to 0} J_1'' \cdot t^{-p} = 0. \tag{3.4.14}$$

Further, by virtue of (3.4.12) the substitution $\xi = y(t) + 2a\zeta\sqrt{t}$ gives

$$\underline{J} = (2a)^{n_0+\epsilon} b_1 \frac{2}{\sqrt{\pi}} \int_{-c}^{d} \left(\frac{y(t)}{2a\sqrt{t}} + \zeta \right)^{n_0+\epsilon} \exp(-\zeta^2)\, d\zeta \cdot t^{(n_0+\epsilon)/2-p}$$

(3.4.15)

$$< J_1' \cdot t^{-p}$$

$$< (2a)^{n_0-\epsilon} b_2 \frac{2}{\sqrt{\pi}} \int_{-c}^{d} \left(\frac{y(t)}{2a\sqrt{t}} + \zeta \right)^{n_0-\epsilon} \exp(-\zeta^2)\, d\zeta t^{(n_0+\epsilon)/2-p} = \overline{J}.$$

Here

$$c = \frac{y(t)}{2a\sqrt{t}}; \qquad d = \frac{\delta - y(t)}{2a\sqrt{t}}. \tag{3.4.15*}$$

Passing now to the limit for $t \to 0$, we obtain

$$\lim_{t\to 0} t^{(n_0+\epsilon)/2-p} (2a)^{n_0+\epsilon} b_1 \frac{2}{\sqrt{\pi}} \int_{-\gamma/a}^{\infty} \left(\frac{\gamma}{a} + \zeta \right)^{n_0+\epsilon} \exp(-\zeta^2)\, d\zeta$$

$$\leqq \lim_{t\to 0} t^{-p} J_1' \tag{3.4.16}$$

$$\leqq \lim_{t\to 0} t^{(n_0-\epsilon)/2-p} (2a)^{n_0-\epsilon} b_2 \frac{2}{\sqrt{\pi}} \int_{-\gamma/a}^{\infty} \left(\frac{\gamma}{a} + \zeta \right)^{n_0-\epsilon} \exp(-\zeta^2)\, d\zeta.$$

Comparing (3.4.14) and (3.4.16) with (3.4.11), we see that

$$\lim_{t\to 0} t^{-(n_0+\epsilon)/2} v_2(t) \geqq (2a)^{n_0+\epsilon} b_1 \frac{2}{\sqrt{\pi}} \int_{-\gamma/a}^{\infty} \left(\frac{\gamma}{a} + \zeta \right)^{n_0+\epsilon} \exp(-\zeta^2)\, d\zeta;$$

(3.4.17)

$$\lim_{t\to 0} t^{-(n_0+\epsilon)/2} v_2(t) \leqq (2a)^{n_0-\epsilon} b_2 \frac{2}{\sqrt{\pi}} \int_{-\gamma/a}^{\infty} \left(\frac{\gamma}{a} + \zeta \right)^{n_0+\epsilon} \exp(-\zeta^2)\, d\zeta.$$

The first of these inequalities signifies that

$$\frac{n_0+\epsilon}{2} \in B_2 \tag{3.4.18}$$

which disproves the assertion that the class B_2 is empty.

Thus we have proved that the divison (A_i, B_i) is a cut. But from the definition of (A_i, B_i) it follows that if α_i is the cut value determined by it then

$$v_i(t) = \alpha_i(t)\, t^{\lambda_i} \qquad (i = 1, 2); \tag{3.4.19}$$

moreover for any $\epsilon > 0$

$$\lim_{t\to 0} \alpha_i(t)\, t^{\lambda_i+\epsilon} = 0; \qquad \lim_{t\to 0} \alpha_i(t)\, t^{\lambda_i-\epsilon} = \infty. \tag{3.4.20}$$

We will now examine $v_1(t)$, continuing to assume that

$$f_1(0) = 0; \qquad f_1'(0) \leqq 0. \tag{3.4.21}$$

We first note that condition (3.4.21) is necessary and sufficient in order that

$$\gamma = \lim_{t \to 0} \sqrt{t} y(t) = 0. \tag{3.4.22}$$

In fact, if $f_1(0) = 0$, then (3.3.33) reduces to the equation

$$\alpha_1 = \alpha_1(1 - 2\operatorname{erf}\gamma), \tag{3.4.23}$$

having a solution $\alpha_1 \neq 0$ only for $\gamma = 0$. But if $\alpha_1 = 0$, then the Stefan condition (3.3.2) in the limit for $t \to 0$ gives

$$\gamma = \lim_{t \to 0} \sqrt{t}\dot{y}(t) = \lim_{t \to 0} \sqrt{t} v_1(t) - \lim_{t \to 0} \sqrt{t} v_2(t) = \alpha_1 - \alpha_2 = -\alpha_2 < 0. \tag{3.4.24}$$

Since $\gamma < 0$ is impossible, we find that

$$\gamma = \alpha_2 = 0, \text{ if } \alpha_1 = 0. \tag{3.4.25}$$

On the other hand, assuming $\gamma = 0$ implies, by virtue of (3.3.33)

$$f_1(0) = 0. \tag{3.4.26}$$

which completes the proof.

We now investigate the auxiliary functions

$$\begin{aligned} \bar{u}(x_1 t) &= f_1(t)\left[1 - \frac{\operatorname{erf}\dfrac{x}{2\sqrt{t}}}{\operatorname{erf}\dfrac{y}{2\sqrt{t}}} + \frac{\operatorname{erf}^2\dfrac{x}{2\sqrt{t}} - \operatorname{erf}^2\dfrac{y}{2\sqrt{t}}}{2\operatorname{erf}\dfrac{y}{2\sqrt{t}}}\right]; \\ \underline{u}(x_1 t) &= f_1(t)\left[1 - \frac{\operatorname{erf}\dfrac{x}{2\sqrt{t}}}{\operatorname{erf}\dfrac{y}{2\sqrt{t}}} - \frac{\operatorname{erf}^2\dfrac{x}{2\sqrt{t}} - \operatorname{erf}^2\dfrac{y}{2\sqrt{t}}}{2\operatorname{erf}\dfrac{y}{2\sqrt{t}}}\right]. \end{aligned} \tag{3.4.27}$$

We have

$$\begin{aligned} &u_1(0,t) = f_1(t) \leqq f_1(t)\left[1 - \frac{1}{2}\operatorname{erf}\frac{y}{2\sqrt{t}}\right] = \bar{u}(0,t) \leqq 0; \\ &\underline{u}(0,t) = f_1(t)\left[1 + \frac{1}{2}\operatorname{erf}\frac{y}{2\sqrt{t}}\right] \leqq f_1(t) = u_1(0,t); \\ &\underline{u}(y(t),t) = u(y(t),t) = \bar{u}(y(t),t) = 0. \end{aligned} \tag{3.4.28}$$

In addition, for $t > 0$ sufficiently small and $0 < x < y(t)$

$$L_1(\bar{u}) \leqq 0; \qquad L_1(\underline{u}) \geqq 0. \tag{3.4.29}$$

In fact,

$$L_1(\bar{u}) = \frac{f_1}{\zeta}\left[z_x^2 - \frac{z}{\zeta}\left(1 - \frac{z}{2}\right)\zeta_t - \frac{\dot{f}_1}{f_1}\left(1 - \frac{z}{\zeta} + \frac{z^2 - \zeta^2}{2\zeta}\right)\zeta + \frac{\zeta\zeta_t}{2}\right], \tag{3.4.30}$$

where

$$z = \operatorname{erf}\frac{x}{2\sqrt{t}}; \qquad \zeta = \operatorname{erf}\frac{y}{2\sqrt{t}}. \tag{3.4.31}$$

Since by assumption

$$\lim_{t\to 0}\sqrt{t}\dot{y}(t) = 0; \qquad \frac{d\ln f}{dt} < kt^{-1}; \quad f_1(t) < 0 \text{ for } t > 0 \tag{3.4.32}$$

we see that for $t > 0$ sufficiently small we have in fact $L_1(\bar{u}) \leqq 0$. Similarily we can verify the second of the equations (3.4.29). But from (3.4.28), (3.4.33) and the maximum principle it follows that

$$\underline{u} \leqq u_1 \leqq \bar{u}. \tag{3.4.33}$$

Hence the last group of equations (3.4.28) implies that

$$\begin{aligned} \frac{\partial}{\partial x}\bar{u}\Big|_{x=y(t)} &= -\frac{f_1(t)}{\sqrt{\pi t}}\frac{\exp\left(-\dfrac{y^2(t)}{4t}\right)}{\operatorname{erf}\dfrac{y(t)}{2\sqrt{t}}}\left[1 - \operatorname{erf}\ \frac{y(t)}{2\sqrt{t}}\right] \leqq v_1(t) \\ &\leqq -\frac{f_1(t)}{\sqrt{\pi t}}\frac{\exp\left(-\dfrac{y^2(t)}{4t}\right)}{\operatorname{erf}\dfrac{y(t)}{2\sqrt{t}}}\left[1 + \operatorname{erf}\ \frac{y(t)}{2\sqrt{t}}\right] = \frac{\partial}{\partial x}\underline{u}\Big|_{x=y(t)} \end{aligned} \tag{3.4.34}$$

Multiplying (3.4.34) by $y(t)\ f^{-1}(t)$ and passing to the limit for

$$\lim_{t\to 0} y(t)\cdot t^{-1/2} = 0, \tag{3.4.35}$$

we find that

$$\lim_{t\to 0}\frac{y(t)\,v_1(t)}{f_1(t)} = -1.^{4)} \tag{3.4.36}$$

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[4] In this regard, see the historical survey and formula (0.1.9) in the Introduction.

But this, by virtue of $(3.2.2_1)$, (3.4.21), (3.4.19) and (3.4.20) implies that

$$y(t) = \mu(t)\, t^{m_0-\lambda_1+1} = \mu(t)\, t^{\nu_0+1}, \tag{3.4.37}$$

where for any $\epsilon > 0$

$$\lim_{t\to 0} \mu(t)\, t^{\epsilon} = 0; \qquad \lim_{t\to 0} \mu(t)\, t^{-\epsilon} = \infty. \tag{3.4.38}$$

Obviously (3.4.37) is equivalent to the relation

$$\dot{y}(t) = \gamma(t)\, t^{\nu_0}; \qquad \nu_0 = m_0 - \lambda_1, \tag{3.4.39}$$

where for any $\epsilon > 0$

$$\lim_{t\to 0} \gamma(t)\, t^{\epsilon} = 0; \qquad \lim_{t\to 0} \gamma(t)\, t^{-p} = \infty. \tag{3.4.40}$$

It is easily seen that

$$\nu_0 \geqq \frac{m_0}{2} > -\frac{1}{2}. \tag{3.4.41}$$

In fact, let us assume the opposite, i.e. that

$$\nu_0 < \frac{m_0}{2}. \tag{3.4.42}$$

By virtue of (3.4.37) this means that

$$\lambda_1 > \nu_0. \tag{3.4.43}$$

Set

$$\lambda_1 - \nu_0 = 2\epsilon > 0. \tag{3.4.44}$$

From (3.4.39) and (3.4.19) it follows that

$$\lim_{t\to 0} \left| \frac{\dot{y}(t)}{v_1(t)} \right| = \lim_{t\to 0} \left| \frac{\gamma_1(t)}{\alpha_1(t)} \right| t^{\nu_0-\lambda_1} = \lim_{t\to 0} \left| \frac{\gamma(t)\, t^{-\epsilon}}{\alpha_1(t)\, t^{\epsilon}} \right| = \infty, \tag{3.4.45}$$

which is impossible, since by virtue of the Stefan condition and the requirement $y(t) > 0$ the inequality

$$0 < \dot{y}(t) \leqq v_1(t) \tag{3.4.46}$$

is valid on a set of positive measure in any neighborhood of $t = 0$.

We now examine $v_2(t)$. From (3.4.37) and the definition $(3.4.24_2)$ it follows that

$$\lim_{t\to 0} \frac{J_2(t)}{v_2(t)} = 0. \tag{3.4.47}$$

In fact, let us choose $\epsilon > 0$ so small that

$$\nu_0 - 3\epsilon > -\frac{1}{2}, \tag{3.4.48}$$

and define $\delta > 0$, $\beta_1 > 0$, $\beta_2 > 0$ and $\mu > 0$ by the conditions

$$\begin{aligned} &0 \leqq \beta_1 t^{\lambda_2+\epsilon} \leqq v_2(t) \leqq \beta_2 t^{\lambda_2-\epsilon}; \\ &0 \leqq |y(t) - y(\tau)| \leqq \mu |t^{1+\nu_0-\epsilon} - \tau^{1+\nu_0-\epsilon}| \quad \text{for} \quad 0 \leqq \tau \leqq t \leqq \delta. \end{aligned} \tag{3.4.49}$$

For such a choice of δ, we find that

$$\begin{aligned} \left| \frac{J_2(t)}{v_2(t)} \right| &\leqq \frac{1}{2a\sqrt{\pi}} \int_0^t \frac{v_2(\tau)}{v_2(t)} \frac{|y(t) - y(\tau)|}{(t-\tau)^{3/2}} \exp\left(-\frac{(y(t) - y(\tau))^2}{4a^2(t-\tau)} \right) d\tau \\ &\leqq \frac{\beta_2 \mu}{2a\beta_1\sqrt{\pi}} \int_0^t \frac{\tau^{\lambda_2-\epsilon}}{t^{\lambda_2+\epsilon}} \frac{t^{1+\nu_0-\epsilon} - \tau^{1+\nu_0-\epsilon}}{(t-\tau)^{3/2}} d\tau \\ &= \frac{\beta_2 \mu t^{\nu_0+1/2-3\epsilon}}{2a\beta_1\sqrt{\pi}} \int_0^1 \frac{z^{\lambda_2-\epsilon}(1 - z^{1+\nu_0-\epsilon})\, dz}{(1-z)^{3/2}}. \end{aligned} \tag{3.4.50}$$

Passing to the limit for $t \to 0$ and taking (3.4.48) into account now shows the validity of (3.4.47). Comparing (3.4.47), ($3.3.24_2$), (3.3.25) and (3.3.23), we again find that

$$\lim_{t\to 0} v_2 t^{-p} = \lim_{t\to 0} v_2 \left(1 - \frac{J_2}{v_2}\right) t^{-p} = \lim_{t\to 0} (J_0 + J_1)\, t^{-p}. \tag{3.4.51}$$

But this shows that the estimate (3.4.17) remains valid for $\gamma = 0$. Comparing this estimate with the definition (3.4.19), we see that

$$\lambda_2 = n_0/2 \quad \text{for} \quad \phi(0) = 0. \tag{3.4.52}$$

Since (3.3.22) is valid for $\gamma = 0$, we have

$$\lambda_2 = -\frac{1}{2} \quad \text{for} \quad \phi(0) \neq 0. \tag{3.4.53}$$

Finally, from $y(t) > 0$, (3.4.19) and the Stefan condition, it follows that

$$\lambda_1 \leqq \lambda_2; \qquad \nu_0 \geqq \lambda_1. \tag{3.4.54}$$

Comparing (3.4.52)—(3.4.54) and (3.4.39) completes the proof of assertions ($3.2.4_2$)—($3.2.4_4$) of Theorem 11.

Let us now prove the last part of this theorem. We note first that if the second of the conditions ($3.2.5_1$) is valid, then it is possible to set $b_1 = b_2 = b_0$

in the inequalities (3.4.17) and go to the limit for $\epsilon \to 0$. Since by assumption it follows from $f_1(0) = 0$ that $\gamma = 0$, we find that

$$\alpha_2 = (2a)^{n_0} b_0 \pi^{-1/2} \Gamma\left(\frac{n_0+1}{2}\right) \quad \text{for} \quad \phi(0) = 0. \tag{3.4.55}$$

At the same time, from (3.3.22) it follows that

$$\alpha_2 = \frac{\phi(0)}{a\sqrt{\pi}} \quad \text{for} \quad \phi(0) \neq 0; \quad \gamma = 0. \tag{3.4.56}$$

Further, integrating $(3.2.2_1)$ and taking relation $(3.2.5_1)$ into account, we obtain

$$f_1(t) = -\psi_1^*(t)\, t^{m_0+1}; \quad \lim_{t\to 0} \psi_1^* = \frac{c_0}{m_0+1}. \tag{3.4.57}$$

Hence from (3.4.36), (3.4.37) and (3.4.19) it follows that

$$\lim_{t\to 0} \alpha_1(t)\, \mu(t) = \frac{c_0}{m_0+1}. \tag{3.4.58}$$

Thus it is possible to have the following cases:

$$\lim_{t\to 0} \alpha_1(t) = \alpha_1 \neq 0, \quad \lim_{t\to 0} \mu(t) = \mu \neq 0; \tag{3.4.59$_1$}$$

$$\lim_{t\to 0} \alpha_1(t) = \infty, \quad \lim_{t\to 0} \mu(t) = 0; \tag{3.4.59$_2$}$$

$$\lim_{t\to 0} \alpha_1(t) = 0, \quad \lim_{t\to 0} \mu(t) = \infty. \tag{3.4.59$_3$}$$

To these conditions we must add the implication of (3.4.37) and (3.4.39) in the form of the equation

$$\lim_{t\to 0} \mu(t) = \lim_{t\to 0} \frac{\gamma(t)}{\nu_0+1}. \tag{3.4.60}$$

On the other hand, in our notation the Stefan condition has the form

$$\gamma(t)\, t^{\nu_0} = \alpha_2(t)\, t^{\lambda_1} - \alpha_2(t)\, t^{\lambda_2}, \tag{3.4.61}$$

where only the following cases are possible:

$$\nu_0 = m_0 - \lambda_1 > \lambda_1, \quad \lambda_1 = \lambda_2; \tag{3.4.62$_1$}$$

$$\nu_0 = \lambda_1 = \lambda_2 = m_0/2; \tag{3.4.62$_3$}$$

$$\nu_0 = \lambda_1 = m_0/2, \quad \lambda_1 < \lambda_2. \tag{3.4.62$_3$}$$

In the first case (3.4.61) gives

$$\lim_{t\to 0}\alpha_1(t) = \lim_{t\to 0}\alpha_2(t) \neq 0, \tag{3.4.63}$$

which by virtue of (3.4.58) and $(3.4.59_1)$ implies

$$\lim_{t\to 0}\gamma(t) = \gamma = \frac{c_0}{\alpha_2}\frac{m_0 - \lambda_2 + 1}{m_0 + 1}. \tag{3.4.64}$$

Equation $(3.4.62_2)$ signifies one of the three cases

$$(3.4.65_1)\quad \lim_{t\to 0}\gamma(t) = \gamma \neq 0, \qquad \lim_{t\to 0}\alpha_1(t) = \alpha_1 > \lim_{t\to 0}\alpha_2(t) = \alpha_2;$$

$$(3.4.65_2)\quad \lim_{t\to 0}\gamma(t) = 0, \qquad \lim_{t\to 0}\alpha_1(t) = \lim_{t\to 0}\alpha_2(t) = \alpha_2;$$

$$(3.4.65_3)\quad \lim_{t\to 0}\gamma(t) = \lim_{t\to 0}\alpha_1(t) = \infty, \quad \lim_{t\to 0}(\gamma(t) - \alpha_1(t)) = -\lim_{t\to 0}\alpha_2(t) = -\alpha_2.$$

But $(3.4.65_2)$ and $(3.4.65_3)$ are incompatible with $(3.4.59_i)$ $(i = 1, 2, 3)$ and (3.4.60). Consequently $(3.4.65_1)$ holds, which in comparison with (3.4.60), (3.4.58) and (3.4.36) gives

$$\gamma = \alpha_1 - \alpha_2; \qquad \frac{\alpha_1\gamma}{m_0 + 2} = \frac{c_0}{2(m_0 + 1)}, \tag{3.4.66}$$

i.e.

$$\gamma = -\frac{\alpha^2}{2} + \sqrt{\frac{\alpha_2^2}{4} + \frac{c_0}{2}\frac{m_0 + 2}{m_0 + 1}}. \tag{3.4.67}$$

Finally, $(3.4.62_3)$ implies

$$\lim_{t\to 0}\alpha_1(t) = \lim_{t\to 0}\gamma(t) = \gamma, \tag{3.4.68}$$

which by virtue of (3.4.58) and (3.4.60) gives

$$\gamma = \sqrt{\frac{c_0(m_0 + 2)}{2(m_0 + 1)}}. \tag{3.4.69}$$

Comparing (3.4.55), (3.4.56), (3.4.52), (3.4.53), (3.4.64), (3.4.67), (3.4.69) with assertions $(3.2.5_3)$ and $(3.2.5_4)$ shows the validity of the latter, which completes the proof of Theorem 11.

§5. Proof of Theorem 12

Before passing to the proof of Theorem 12, we note that inconsistency of the boundary and initial conditions of the problem signifies that for some period of time the medium is in a one-phase state, and

consequently its temperature at the boundary $x = 0$ is for some period nonnegative.[5] Therefore, inconsistency of preassigned conditions in the third boundary problem of Stefan can be derived immediately by investigating the corresponding boundary problem for the second phase on the interval $[0, 1]$. However, we do not follow this course, but deduce the inconsistency conditions incidentally, in the process of determining the initial speed of propagation of the interphase boundary in the original Stefan problem.

We will not give the proof of Theorem 12 in every detail, but will limit ourselves to exhibiting the method of proof and the basic intermediate results. Every intermediate calculation can be done without difficulty.

Turning to the proof of Theorem 12, we show that if u_1, u_2, y is a solution of problem A_3, and v_1 and v_2 are defined, as everywhere above, by relations (2.1.7), then in the case $y(0) = l = 0$, $v(t)$ is a solution to the integral equation

$$
\begin{aligned}
v_1(t) = &-2h\int_0^t f_1(\tau)\frac{\partial}{\partial\xi}g_4(y(t),0,t-\tau)\,d\tau \\
&-2\int_0^t v_1(\tau)\frac{\partial}{\partial\xi}g_4(y(t),y(\tau),t-\tau)\,d\tau,
\end{aligned}
\tag{3.5.1}
$$

where g_4 is defined in agreement with (0.2.10), $v_2(t)$ is a solution of equation $(3.1.1_2)$ and $y(t)$ satisfies the Stefan condition

$$
\dot{y}(t) = v_1(t) - v_2(t); \quad y(0) = 0. \tag{3.5.2}
$$

As in the proof of Theorem 11, the investigation of this equation allows us to conclude that v_i and $\dot{y}$ are given by the equations

$$
\begin{aligned}
v_i(t) &= \alpha_i t^{\lambda_i}(1 + v_i^*(t)) \qquad (i = 1, 2); \\
\dot{y}(t) &= \gamma t^{r}(1 + y_1^*(t)),
\end{aligned}
\tag{3.5.3$_1$}
$$

where

$$
\lim_{t\to 0} v_i^*(t) = \lim_{t\to 0} y_1^*(t) = 0. \tag{3.5.3$_2$}
$$

Hence the following relation remains valid:

---

[5] For convenience we use the terminology of the thermal problem.

(3.5.4*)
$$\lambda_2 = \begin{cases} -\frac{1}{2} & \text{for } \phi(0) \neq 0 \\ \frac{n_0}{2} & \text{for } \phi(0) = 0; \end{cases}$$

(3.5.4)
$$\alpha_2 = \begin{cases} \dfrac{\phi(0)}{a\sqrt{\pi}} \dfrac{\exp\left(-\dfrac{\gamma^2}{a^2}\right)}{\operatorname{erfc}\dfrac{\gamma}{a}} & \text{for } \nu = -\frac{1}{2};\quad \phi(0) \neq 0 \\ \dfrac{\phi(0)}{a\sqrt{\pi}} & \text{for } \nu > -\frac{1}{2};\quad \phi(0) \neq 0 \\ (2a)^{n_0} b_0 \Gamma\left(\dfrac{n_0+1}{2}\right)\dfrac{1}{\sqrt{\pi}} & \text{for } \nu > -\frac{1}{2};\quad \phi(0) = 0. \end{cases}$$

We investigate v_1 by exactly the same method as that used for v_2. By virtue of (3.5.1) and (0.2.10) we have

(3.5.5)
$$v_1(t) = I_1 + J_1 + I_2 + J_2,$$

where

(3.5.6$_1$)
$$\lim_{t\to 0} \frac{J_j}{I_j} = 0 \quad (j = 1, 2);$$

$$I_1 = -\frac{h}{\sqrt{\pi}} \int_0^t f_1(\tau) \frac{y(t)}{(t-\tau)^{3/2}} \exp\left(-\frac{y^2(t)}{4(t-\tau)}\right) d\tau;$$

(3.5.6$_2$)
$$I_2 = -\frac{1}{2\sqrt{\pi}} \int_0^t v_1(\tau) \left\{ \frac{y(t)-y(\tau)}{(t-\tau)^{3/2}} \exp\left(-\frac{(y(t)+y(\tau))^2}{4(t-\tau)}\right) + \frac{y(t)+y(\tau)}{(t-\tau)^{3/2}} \exp\left(-\frac{(y(t)+y(\tau))^2}{4(t-\tau)}\right) \right\} d\tau.$$

It is easily shown that for arbitrary real p

(3.5.7)
$$\lim_{t\to 0} I_1 \cdot t^{-p} = \lim_{t\to 0} t^{m_0 - p} \cdot 2hc_0 j;$$
$$\lim_{t\to 0} I_2 \cdot t^{-p} = -\alpha_1^* \lim t^{\lambda_1 - p},$$

where

$$j = \begin{cases} \dfrac{2}{\sqrt{\pi}} \displaystyle\int_\gamma^\infty \left(1 - \frac{\gamma^2}{z^2}\right)^{m_0} \exp(-z^2)\, dz & \text{for } \nu = -\frac{1}{2} \\ 1 & \text{for } \nu > -\frac{1}{2}; \end{cases}$$

(3.5.8)

$$\alpha_1^* = \begin{cases} \alpha_1 \text{ for } \nu > -\dfrac{1}{2} \\ \alpha_1 \left[1 - 2\operatorname{erf}\gamma + \dfrac{\gamma}{\sqrt{\pi}} \displaystyle\int_0^1 \frac{\exp\left(-\gamma^2 \dfrac{1-\sqrt{z}}{1+\sqrt{z}}\right) - \exp\left(-\gamma^2 \dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{\sqrt{z}\sqrt{1-z}}\, dz \right] \\ \qquad \text{for } \nu = -\dfrac{1}{2}. \end{cases}$$

Comparing $(3.5.6_1)$, (3.5.7) and $(3.5.3_i)$ $(i = 1, 2)$ and assuming that $p = \lambda_1$, we find that

$$\alpha_1 = \begin{cases} A(\gamma) \lim\limits_{t \to 0} t^{m_0 - \lambda_1} & \text{for } \nu = -\dfrac{1}{2} \\ hc_0 \lim\limits_{t \to 0} t^{m_0 - \lambda_1} & \text{for } \nu > -\dfrac{1}{2} \end{cases}, \tag{3.5.9}$$

where

$$A(\gamma) > 0. \tag{3.5.9*}$$

Since by definition $\alpha_1 > 0$, we see that

$$\lambda_1 = m_0. \tag{3.5.10}$$

Comparing (3.5.9), (3.5.10) and (3.5.4) with the condition

$$\gamma \lim_{t \to 0} t^{r - \min \lambda_i} = \alpha_1 \lim_{t \to 0} t^{\lambda_1 - \min \lambda_i} - \alpha_2 \lim_{t \to 0} t^{\lambda_2 - \min \lambda_i} \tag{3.5.11}$$

implied by (3.5.2), we find that for compatible boundary conditions of the problem it is still necessary that the conditions $(3.2.6_1')$ and $(3.2.6_1'')$ of the theorem remain valid. But in this case the comparison implies the validity of $(3.2.6_2)$ and $(3.2.6_3)$, which proves the theorem.

CHAPTER IV

SOLUTION OF PROBLEMS A_1 AND A_3 [131], [143]

§1. The first boundary problem (A_1). Subsidiary functions

We turn now to the proof of existence of a solution to problem A_1. Here the fact that the initial domain of existence of one of the phases degenerates to a point destroys the convergence of the Picard iteration process. Problem A_1 is not solved as easily as problem B_1, or the problem investigated above in Chapter I. What is more, the method which we have available allows the solution of problem A_1 only under certain additional limitations on the initial and boundary conditions of the problem, namely for those limitations which are imposed in Theorem 11. These limitations undoubtedly are connected not with the existence problem but only with the method chosen for its solution.[1)]

For simplicity we will assume the validity of condition $(3.2.5_1)$ of Theorem 11. In addition we shall assume that

$$
\begin{aligned}
&f_1(0) = \phi(0) = 0; \quad \dot{f}_1(t) = -c_0 t^{m_0}(1 + f_1^*(t)); \\
&\dot{\phi}(x) = b_0 x^{n_0}(1 + \phi^*(x)); \\
&m_0 \geqq 0; \quad n_0 \geqq 0; \quad |f_1^*(t)| \leqq Mt, \quad t \geqq 0; \\
&|\phi^*(x)| \leqq Mx; \quad 0 \leqq x \leqq 1.
\end{aligned} \tag{4.1.1}
$$

The solution of problem A_1 will be constructed as a limit of a sequence of series solutions of a problem similar to problem B_1. The formulation and proof of the corresponding result is given by means of an auxiliary construction which will be used below.

As follows from the proof of Theorem 11, we have the a priori estimate

---

[1)] We are speaking here of the construction of a "classical" solution to the problem. As noted in §1 of the Introduction, the generalized solution of the problem was constructed by S. L. Kamenomostskaja [73] and O. A. Oleĭnik [107] for very general cases. Their results will be stated briefly below (see Chapter VII, §2).

$$v_i(t) = \alpha_i t^{\lambda_i}(1 + v_i^*(t)) \quad (i = 1, 2); \quad \dot{y}(t) = \gamma t^{\nu}(1 + y_1^*(t)), \tag{4.1.2}$$

where

$$\lim_{t \to 0} v_i^* = \lim_{t \to 0} y_1^* = 0 \tag{4.1.2*}$$

and α_i, λ_i, γ and ν are defined by the conditions

$$\begin{aligned} &\lambda_1 + \nu = m_0; \quad \lambda_2 = \frac{n_0}{2}; \quad \lambda_1 \leqq \lambda_2; \\ &\alpha_1 = \alpha_2 \text{ for } m_0 > n_0; \quad \alpha_1 = \gamma \text{ for } m_0 < n_0; \\ &\gamma = \alpha_1 - \alpha_2 \text{ for } m_0 = n_0; \\ &\alpha_2 = (2a)^{n_0} b_0 \Gamma\left(\frac{n_0 + 1}{2}\right)\frac{1}{\sqrt{\pi}}; \quad \alpha_1 \gamma = c_0 \frac{\nu + 1}{m_0 + 1}. \end{aligned} \tag{4.1.3}$$

Now set

$$\begin{aligned} &z_1(t) = \frac{\gamma t^{\nu+1}}{\nu + 1}(1 - \delta\sqrt{t}); \quad z_2(t) = \frac{\gamma t^{\nu+1}}{\nu + 1}(1 + \delta\sqrt{t}); \\ &w_{1,1} = f_1(t)\left[1 - \frac{\operatorname{erf}(x/2\sqrt{t})}{\operatorname{erf}(z_1/2\sqrt{t})} + \frac{\operatorname{erf}^2(x/2\sqrt{t}) - \operatorname{erf}^2(z_1/2\sqrt{t})}{2\operatorname{erf}(z_1/2\sqrt{t})}\right]; \\ &w_{1,2} = f_1(t)\left[1 - \frac{\operatorname{erf}(x/2\sqrt{t})}{\operatorname{erf}(z_1/2\sqrt{t})}\right], \end{aligned} \tag{4.1.4}$$

and define $w_{2,j}$ $(j = 1, 2)$ as solutions of the problem

$$\begin{aligned} &L_2(w_{2,j}) = 0 \text{ for } z_j(t) < x < 1; \quad t > 0; \\ &w_{2,j} = 0 \text{ for } x = z_j(t); \quad w_{2,j} = f_2 \text{ for } x = 1; \quad t > 0; \\ &w_{2,j} = \phi \text{ for } t = 0; \quad 0 < x < 1. \end{aligned} \tag{4.1.5}$$

It is easy to see that there exists some $\delta > 0$ and $T_0 > 0$ such that for $0 \leqq t \leqq T_0$

$$L_1(w_{1,1}) \leqq 0 \text{ for } 0 < x < z_1(t); \quad L_1(w_{1,2}) \geqq 0 \text{ for } 0 < x < z_2(t) \tag{4.1.6}$$

and simultaneously

$$\left.\begin{aligned} \dot{z}_1(t) &< v_{1,1}(t) - v_{2,1}(t) \\ \dot{z}_2(t) &> v_{1,2}(t) - v_{2,2}(t) \end{aligned}\right\} \quad 0 \leqq t \leqq T. \tag{4.1.7}$$

Here

$$v_{i,j}(t) = \frac{\partial}{\partial x} w_{i,j} \text{ for } x = z_j(t) \qquad (i, j = 1, 2). \tag{4.1.7*}$$

In fact, let $\delta > 0$ be fixed. Then (4.1.6) is valid for $0 \leqq t \leqq T_0$ where T_0 is sufficiently small, as is implied from the definition (4.1.4) and condition (4.1.1). Thus it is only necessary to prove (4.1.7). We will show that

$$z_j(t) < 2\sqrt{t} \text{ for } 0 \leqq t \leqq T_0 \quad (j = 1, 2). \tag{4.1.8}$$

We have

$$\begin{aligned} v_{1,1} &= -\frac{f_1(t)}{\sqrt{\pi t}} \frac{\exp(-z_1^2(t)/4t)}{\operatorname{erf}(z_1(t)/2\sqrt{t})} \left[1 - \operatorname{erf}\frac{z_1(t)}{2\sqrt{t}}\right]; \\ v_{1,2} &= -\frac{f_1(t)}{\sqrt{\pi t}} \frac{\exp(-z_2^2(t)/4t)}{\operatorname{erf}(z_2(t)/2\sqrt{t})}. \end{aligned} \tag{4.1.9}$$

Estimating $v_{1,1}$ from below and $v_{1,2}$ from above, we find with the aid of (4.1.1), (4.1.8) and (4.1.4) that

$$\begin{aligned} v_{1,1}(t) &> \alpha_1 t^{\lambda_1} \left[1 - \frac{\gamma t^{\nu+1/2}}{(\nu+1)\sqrt{\pi}}\right]\left[1 - \frac{\gamma^2 t^{2\nu+1}}{4(\nu+1)^2}\right] \frac{1 - Mt}{1 - \delta\sqrt{t}}; \\ v_{1,2}(t) &< \alpha_1 t^{\lambda_1} \frac{1 + Mt}{1 + \delta\sqrt{t}}. \end{aligned} \tag{4.1.10}$$

We now examine $v_{2,j}(t)$. The equation defining $v_{2,j}(t)$ can be written in the form

$$v_{2,j}(t) = I_{1,j}(t) + I_{2,j}(t) + J_j(t), \tag{4.1.11}$$

where $I_{i,j}$ is obtained from the expressions J_i $(i = 1, 2)$ defined in $(3.3.24_2)$ by replacing v_2 by $v_{2,j}$ and y by z_j, by taking J_j as J_3 in $(3.3.24_2)$ and by adding the integrals tending to zero as an arbitrary power of t, so that for any $p > 0$ there is some α_p such that

$$|J_j(t)| < \alpha_p t^p \text{ for } 0 \leqq t \leqq T \quad (j = 1, 2). \tag{4.1.12}$$

Let

$$\max_{0 \leqq \tau \leqq T_0} v_{2,j}(\tau) = v_{2,j}(t). \tag{4.1.13}$$

Using the theorem on the mean and taking (4.1.4) into account, we find that

$$\begin{aligned} v_{2,1}(t) &< \frac{I_{1,1}(t) + \alpha_p t^p}{1 - (\gamma/a\sqrt{\pi})t^{\nu+1/2}}; \\ v_{2,2}(t) &> I_{1,2}(t) - \alpha_p t^p. \end{aligned} \tag{4.1.14}$$

On the other hand, by virtue of (4.1.1) and the definition $(3.3.24_2)$

we have

$$I_{1,1}(t) < b_0(I^1_{n_0} + MI^1_{n_0+1}); \quad I_{1,2}(t) > b_0(I^2_{n_0} - MI^2_{n_0+1}), \tag{4.1.15}$$

where

$$I^j_m = \frac{1}{a\sqrt{\pi t}} \int_0^1 \xi^m \exp\left(-\frac{(\xi - z_j(t))^2}{4a^2 t}\right) d\xi$$

$$= (2a)^m t^{m/2} \frac{2}{\sqrt{\pi}} \int_{c_j}^{d_j} \left(\zeta + \frac{z_j(t)}{2a\sqrt{t}}\right)^m \exp(-\zeta^2)\, d\zeta; \tag{4.1.15*}$$

$$c_j = -z_j(t) \cdot (2a\sqrt{t})^{-1}; \quad d_j = (1 - z_j(t))(2a\sqrt{t})^{-1}.$$

It is easily seen that

$$(2a)^m t^{m/2} \Gamma\left(\frac{m+1}{2}\right) \frac{1}{\sqrt{\pi}} \left[1 - \frac{\sqrt{\pi}\beta_m}{\Gamma((m+1)/2)} \left(1 - \operatorname{erf} \frac{1 - z_j(t)}{2a\sqrt{t}}\right)\right] < I^j_m$$

$$< (2a)^m t^{m/2} \Gamma\left(\frac{m+1}{2}\right) \frac{1}{\sqrt{\pi}} \left[1 + \frac{z_j(t)}{2a\sqrt{t}} P_{m-1}\left(\frac{z_j(t)}{2a\sqrt{t}}\right)\right], \tag{4.1.16}$$

where P_{m-1} is a polynomial of order $m-1$ and $\beta_m = 2 \max_z z^m \exp(-3z^2/4)$. Using (4.1.16), (4.1.4), (4.1.3) and (4.1.15), we obtain

$$I_{1,1}(t) < \alpha_2 t^{\lambda_2} \left[1 + 2aM\sqrt{t} \frac{\Gamma((n_0+2)/2)}{\Gamma((n_0+1)/2)}\right]$$

$$\cdot \left[1 + \frac{z_1(t)}{2a\sqrt{t}} P_{n_0}\left(\frac{z_1(t)}{2a\sqrt{t}}\right)\right];$$

$$I_{1,2}(t) > \alpha_2 t^{\lambda_2} \left[1 - \frac{\sqrt{\pi}\beta_{n_0}}{\Gamma((n_0+1)/2)} \left(1 - \operatorname{erf} \frac{1 - z_2(t)}{2a\sqrt{t}}\right)\right.$$

$$\left. - 2aM\sqrt{t} \frac{\Gamma((n_0+2)/2)}{\Gamma((n_0+1)/2)} \left(1 + \frac{z_2(t)}{2a\sqrt{t}} P_{n_0}\left(\frac{z_2(t)}{2a\sqrt{t}}\right)\right)\right]. \tag{2.1.17}$$

Substituting (4.1.17) into (4.1.14) and taking into account the fact that in (4.1.14) $p > 0$ is arbitrary, we find that

$$v_{2,1}(t) < \alpha_2 t^{\lambda_2}(1 + L\sqrt{t})(1 + \psi_1(\delta_1 t));$$

$$v_{2,2}(t) > \alpha_2 t^{\lambda_2}(1 - L\sqrt{t})(1 + \psi_2(\delta, t)). \tag{4.1.18}$$

Here

$$\psi_i = N t^{r+1/2} P_i(1 - (-1)^i \delta\sqrt{t}), \tag{4.1.19}$$

with L and N positive constants and $P_i(x)$ a polynomial increasing with x.

Comparing (4.1.10), (4.1.18) and (4.1.4) and taking into account that $\alpha_1 > \alpha_2$ for $\lambda_1 = \lambda_2$, we see that assertion (4.1.7) holds for sufficiently large $\delta > 0$, and sufficiently small $T_0 > 0$.

§2. Existence of a solution to problem A_1

We now choose an arbitrary monotonically decreasing sequence

$$t_0 > t_1 > \cdots > t_n > \cdots; \quad \lim_{n\to\infty} t_n = 0; \quad t_0 < T_0, \tag{4.2.1}$$

where T_0 is defined by conditions (4.1.6) and (4.1.7), and we define $u_{i,n}$, y_n as solutions of problem B_1:

$$\begin{aligned}
&L_1(u_{1,n}) = 0 \text{ for } 0 < x < y_n(t); \quad t > t_n;\\
&L_2(u_{2,n}) = 0 \text{ for } y_n(t) < x < 1; \quad t > t_n;\\
&u_{1,n}(0,t) = f_1(t); \quad u_{i,n}(y_n(t),t) = 0 \quad (i = 1,2);\\
&u_{2,n}(1,t) = f_2(t); \quad t > t_n;\\
&u_{i,n}(x,t_n) = w_{i,1}(x,t_n) \quad (i = 1,2);\\
&\dot{y}_n(t) = v_{1,n}(t) - v_{2,n}(t); \quad y_n(t_n) = z_1(t_n).
\end{aligned} \tag{4.2.2}$$

Here

$$v_{i,n}(t) = \frac{\partial}{\partial x} u_{i,n}(y_n(t),t). \tag{4.2.2*}$$

We now have

THEOREM 14. *The sequences* $\{u_{i,n}(x,t)\}$ *and* $\{y_n(t)\}$ *converge uniformly to the unique solution of problem* A_1.

The uniqueness of the solution to problem A_1 follows from the theorem proved below (see Chapter VII, §2) for the generalized solution of the Stefan problem, developed by S. L. Kamenomostskaja [73] and O. A. Oleĭnik [107], or from the elementary proof of the theorem of uniqueness cited in the end of this section, which is valid under very strict additional limitations on the original data of the problem. Thus it is sufficient to prove Theorem 14 in the following weaker formulation.

THEOREM 14a. *The family* $\{u_{i,n}(x,t)\}$, $\{y_n(t)\}$ *is compact in the spatial norm of uniform convergence. Any uniformly convergent subsequence of functions* $\{u_{i,n_j}(x,t)\}$ $(i = 1,2)$, $\{y_{n_j}(t)\}$ *is convergent to the solution* $u_i(x,t)$ $(i = 1,2)$, $y(t)$ *of problem* A_1.

The proof of Theorem 14a reduces to the proof of the following lemmas.

LEMMA 7. *The sequences* $\{v_{i,n}(t)\}$, $\{y_n(t)\}$ *are defined, uniformly bounded and equicontinuous on an interval* $0 < t < T$ *where* $T > 0$ *is sufficiently small.*

LEMMA 8. *If*

$$y(t) = \lim_{n_j \to \infty} y_{n_j}(t), \tag{4.2.3}$$

then there exists some $T^* > 0$ *so small that for* $0 < t < T^*$

$$\frac{\gamma}{\nu+1} t^{\nu+1}(1 - \delta\sqrt{t}) < y(t) < \frac{\gamma}{\nu+1} t^{\nu+1}(1 + \delta\sqrt{t}); \tag{4.2.4}$$

$$0 < \dot{y}(t) < \gamma t^{\nu}(1 + \delta\sqrt{t}).$$

Here $\delta > 0$ *coincides with that* δ *chosen in the previous section.*

LEMMA 9. *There exists no nontrivial solution to the integral equation*

$$w = -2 \int_0^t w(\tau) \frac{\partial}{\partial \xi} g_2(y(t), y(\tau), t - \tau)\, d\tau. \tag{4.2.5}$$

In fact, assume that Lemma 7—9 are proved. From Lemma 7 and the theorem of Arzelà we obtain the existence of a subsequence $\{n_j\}$ such that the sequences $\{v_{i,n_j}\}$ and $\{y_{n_j}\}$ converge uniformly everywhere in their domains of definition. Let

$$v_i(t) = \lim_{n_j \to \infty} v_{i,n_j}(t); \quad y(t) = \lim_{n_j \to \infty} y_{n_j}(t). \tag{4.2.6}$$

We define $U_i(x,t)$ as the solution of the problem

$$\begin{aligned} &L_1 U_1 = 0,\ 0 < x < y(t); \quad L_2 U_2 = 0,\ y(t) < x < 1; \\ &U_1|_{x=0} = f_1(t); \quad U_i|_{x=y(t)} = 0; \quad U_2|_{x=1} = f_2(t); \\ &U_2|_{t=0} = \phi(x),\ 0 < x < 1. \end{aligned} \tag{4.2.7}$$

It is easily seen that the solution U_1 of this problem exists.[2] For its construction it is sufficient to examine the problem (4.2.7) on the interval $t_0 < t < T$, where $t_0 > 0$ is arbitrarily small, adjoining to it the initial condition

$$U_1(x, t_0) = f_1(t_0)\left(1 - \frac{x}{y(t_0)}\right), \tag{4.2.7*}$$

[2] The existence of U_2 is a trivial fact, since its domain of definition is nontrivial.

and passing to the limit for $t_0 \to 0$ in the solution obtained [50].

Let

$$V_i(t) = \partial U_i(y(t), t)/\partial x \qquad (i = 1, 2). \tag{4.2.8}$$

We will show that

$$V_i(t) = v_i(t) \quad (i = 1, 2), \tag{4.2.9}$$

whence it will obviously follow that U_i, y is a solution of problem A_1, since from

$$\dot{y}_n(t) = v_{1,n}(t) - v_{2,n}(t) \tag{4.2.10*}$$

it follows that

$$\dot{y}(t) = v_1(t) - v_2(t). \tag{4.2.10}$$

From definition (4.2.7) it follows that $V_i(t)$ is a solution of the integral equations $(3.1.1_i)$ $(i = 1, 2)$, in which it is necessary to assume that $l = 0$ and $\phi_2 \equiv \phi$. On the other hand, $v_{i,n}$, y_n is a solution of the system $(2.2.3_i)$, where t_0 is replaced by t_n, l by $z_1(t_n)$ and $\phi_i(x)$ by $w_i(x, t_n)$. Passing to the limit in these equations for $n_j \to \infty$, we find that $v_i(t)$ is an exact solution of them. Since these are equations satisfied by V_i $(i = 1, 2)$, the proof of (4.2.9) is complete if we prove the uniqueness of the solution to the integral equations

$$w_1(t) = -2\int_0^t w_1(\tau)\frac{\partial}{\partial \xi} g_2(y(t), y(\tau), t - \tau)\, d\tau; \tag{4.2.11$_1$}$$

$$w_2(t) = 2a^2 \int_0^t w_2(\tau)\frac{\partial}{\partial \xi} G_2(y(t), y(\tau), t - \tau)\, d\tau. \tag{4.2.11$_2$}$$

But the uniqueness of the solution to equation $(4.2.11_2)$ is a trivial fact, since clearly $y(t) < 1$. The uniqueness of the solution to equation $(4.2.11_1)$ is verified by Lemma 9.

Thus for the proof of Theorem 14a it remains for us only to prove the existence of the limit

$$\lim_{n_j \to \infty} u_{i,n_j}(x, t) = u_i(x, t) \quad (i = 1, 2) \tag{4.2.12}$$

and its coincidence with $U_i(x, t)$. But this follows at once from the representation of U_i and u_{i,n_j} in the form of a quadrature of type $(2.2.4_i)$ $(i = 1, 2)$ and the obvious possibility of taking the limit in them for $n_j \to \infty$.

We now turn to the proof of Lemmas 7—9. We begin by proving Lemma 9 under the assumption that Lemma 8 is proved. By virtue of Lemma 8

$$\dot{y}(t) > 0 \text{ for } 0 < t \leqq T, \tag{4.2.13}$$

where $T > 0$ is sufficiently small. Hence by definition (2.2.2)

$$I = 2\int_0^t \left|\frac{\partial}{\partial \xi} g_2(y(t), y(\tau), t-\tau)\right| d\tau = 1 - 2\operatorname{erf}\frac{y(t)}{2\sqrt{t}} + I^*(t), \tag{4.2.14}$$

where

$$I^* = \frac{1}{2\sqrt{\pi}} \int_0^t \frac{\dot{y}(\tau)}{(t-\tau)^{3/2}} d\tau \int_{y(t)-y(\tau)}^{y(t)+y(\tau)} \zeta \exp\left(-\frac{\zeta^2}{4(t-\tau)}\right) d\zeta. \tag{4.2.14*}$$

Reversing the order of integration and using Lemma 8, we find that

$$\begin{aligned} I^*(t) &< \frac{1}{2\sqrt{\pi}} \int_0^{2y(t)} \zeta d\zeta \int_0^t \frac{\dot{y}(\tau)}{(t-\tau)^{3/2}} \exp\left(-\frac{\zeta^2}{4(t-\tau)}\right) d\tau \\ &< \frac{2\gamma^2 t^{2\nu+1}}{\nu+1} (1 + \delta\sqrt{t})^2 = I_1. \end{aligned} \tag{4.2.15}$$

In addition,

$$2\operatorname{erf}\frac{y(t)}{2\sqrt{t}} > 2\operatorname{erf}\left[\frac{\gamma t^{\nu+1/2}}{2(\nu+1)} (1 - \delta\sqrt{t})\right] = I_0. \tag{4.2.16}$$

From (4.2.15) and (4.2.16) it follows that

$$I < 1 - I_0 + I_1 = \tilde{I}(t). \tag{4.2.17}$$

But

$$\tilde{I}(0) = 1; \quad d\tilde{I}/dt < 0 \tag{4.2.18}$$

for $t > 0$ sufficiently small. Consequently it is possible to find $T > 0$ so small that

$$I < 1 \text{ for } 0 < t < T. \tag{4.2.19}$$

Let $\max|w(\tau)|$ for $0 \leqq \tau \leqq T$ be attained for $\tau = t > 0$. Taking the absolute value of w and using (4.2.14), we find that

$$\begin{aligned} |w(t)| &\leqq 2\int_0^t |w(\tau)| \left|\frac{\partial}{\partial \xi} g_2(y(t), y(\tau), t-\tau)\right| d\tau \\ &\leqq |w(t)| \cdot I < |w(t)|, \end{aligned} \tag{4.2.20}$$

which is impossible if $w(t) \not\equiv 0$; this completes the proof.

We turn to the proof of Lemma 7. We first note that by virtue of the results of Chapter II the sequences $\{u_{i,n}(x,t)\}$, $\{v_{i,n}(t)\}$ and $\{y_n(t)\}$ are defined on some interval $t_n < t < T$, where T is not dependent on n. Further, by virtue of (4.1.4), (4.1.7) and (4.2.2) the functions $w_{i,j}$, z_j and $u_{i,n}$, y_n satisfy all the conditions of Lemma 1 for $t_n \leqq t \leqq T_0$, whatever the value of n. Consequently

$$z_1(t) \leqq y_n(t) \leqq z_2(t) \text{ for } t_n < t < T \quad (n = 1, 2, \cdots). \tag{4.2.21}$$

Moreover, from the definition of $y_n(t)$ it follows that for any n there exists some $\tau_n > t_n$ such that

$$\dot{y}_n(t) > 0 \text{ for } t_n < t < \tau_n. \tag{4.2.22}$$

Let τ_n^* be a precise upper bound on the set of all τ_n satisfying condition (4.2.22). We will show that

$$T^* = \inf_n \{\tau_n^*\} > 0 \quad (n = 0, 1, 2, \cdots). \tag{4.2.23}$$

We first note that from the maximum principle it follows that for all $t \in (t_n, \tau_n^*)$

$$u_{1,n}(x,t) \geqq f_1(t)\left(1 - \frac{x}{y_n(t)}\right) \equiv U_{1,n}^*(x,t), \tag{4.2.24}$$

or

$$\begin{aligned} &L_1 U_{1,n}^* \geqq 0 \text{ for } t_n < \tau < \tau_n^*; \\ &U_{1,n}^*|_{x=0} = u_{1,n}|_{x=0} = f_1(t) < 0; \quad U_{1,n}^*|_{x=y_n(t)} = 0, \end{aligned} \tag{4.2.25}$$

and by virtue of (4.2.2) and (4.1.4)

$$\begin{aligned} u_{1,n}|_{t=t_n} &> f_1(t_n)\left[1 - \frac{\operatorname{erf}(x/2\sqrt{t_n})}{\operatorname{erf}(z_1/2\sqrt{t_n})}\right] > f_1(t_n)\left(1 - \frac{x}{z_1(t_n)}\right) \\ &\geqq f_1(t_n)\left(1 - \frac{x}{y_n(t_n)}\right) = U_{1,n}^*(x, t_n). \end{aligned} \tag{4.2.26}$$

In order to deduce the inequality $L_1 U_{1,n}^* \geqq 0$ we assume that $\dot{f}_1 < 0$ for $0 < t \leqq T_0$, as is possible by virtue of the definition of $f_1(t)$. From (4.2.24) and the fact that $U_{1,n}(y_n(t), t) \equiv 0$ it follows that

$$0 \leqq v_{1,n}(t) \leqq -\frac{f_1(t)}{y_n(t)} \text{ for } t_n \leqq t \leqq \tau_n^*. \tag{4.2.27}$$

Majorizing this inequality by use of (4.2.21), we obtain

$$0 \leqq v_{1,n}(t) \leqq -\frac{f_1(t)}{z_1(t)} \leqq \mu; \quad t_n \leqq t \leqq \tau_n^*, \tag{4.2.28}$$

where $\mu = \text{const.} > 0$.

We now estimate $v_{2,n}(t)$, using the integral equation $(2.2.3_2)$ which in the case under investigation has the form

$$\begin{aligned} v_{2,n}(t) &= 2\int_{t_n}^{t} \dot{f}_2(\tau)\, G_2(y_n(t), 1, t-\tau)\, d\tau \\ &+ 2\int_{y_n(t_n)}^{1} \frac{\partial}{\partial \xi} w_{2,1}(\xi, t_n)\, G_2(y_n(t), \xi, t-t_n)\, d\xi \\ &+ 2a^2 \int_{t_n}^{t} v_{2,n}(\tau) \frac{\partial}{\partial \xi} G_2(y_n(t), y_n(\tau), t-\tau)\, d\tau. \end{aligned} \tag{4.2.29}$$

Repeating the estimate of $v_{2,j}$ from §1, using (4.2.21), and noting that (4.1.5) implies

$$\lim_{t \to 0} \frac{\partial}{\partial x} w_{2,1} = \dot{\phi}(x), \tag{4.2.30}$$

we obtain the estimate

$$v_{2,n}(t) < \alpha_2 t^{\lambda_2}(1 + v_2^*(t)) \text{ for } t_n \leqq t \leqq \tau_n^*, \tag{4.2.31}$$

where $\lim_{t \to 0} v_2^* = 0$, $v_2^* > 0$ is not dependent on n, and α_2 and λ_2 are defined in agreement with (4.1.3).

We now estimate $v_{1,n}(t)$ from below. We set

$$\overline{U}_n = f_1(t) \left[1 - \frac{\operatorname{erf} \dfrac{x}{2\sqrt{t}}}{\operatorname{erf} \dfrac{y_n(t)}{2\sqrt{t}}} + \frac{\operatorname{erf}^2 \dfrac{x}{2\sqrt{t}} - \operatorname{erf}^2 \dfrac{y_n(t)}{2\sqrt{t}}}{2 \operatorname{erf} \dfrac{y_n(t)}{2\sqrt{t}}} \right]. \tag{4.2.32}$$

We have already noted that functions of this type are superparabolic in a neighborhood of $t = 0$, i.e. that

$$L_1(\overline{U}_n) \leqq 0 \text{ for } 0 < t < \tau_n^{**}, \tag{4.2.33}$$

where τ_n^{**} is sufficiently small.

One can estimate the value of τ_n^{**} more accurately. Again using, for simplicity, the notation

$$z = \operatorname{erf} \frac{x}{2\sqrt{t}}; \qquad \zeta = \operatorname{erf} \frac{y_n(t)}{2\sqrt{t}}, \tag{4.2.34}$$

we find that

$$L_1 \overline{U}_n = \frac{f_1(t)}{\zeta}\left\{ z_x^2 + \zeta_t\left[\frac{\zeta}{2} - \frac{z}{\zeta}\left(1 - \frac{z}{2}\right)\right] - \frac{\dot{f}_1(t)}{f_1(t)}\zeta\left(1 - \frac{z}{\zeta} - \frac{\zeta^2 - z^2}{2\zeta}\right)\right\}. \tag{4.2.35}$$

Majorizing (4.2.35) with the help of (4.2.21) and (4.2.28), we obtain

$$\begin{aligned} L_1 \overline{U}_n &\leqq \frac{f_1(t)}{\zeta(t)}\left[\frac{\exp(-z_2^2/2t)}{\pi t} - \frac{\mu \exp(-z_1^2/4t)}{\sqrt{\pi t}} - \frac{M_0}{t}\operatorname{erf}\frac{z_2(t)}{2\sqrt{t}}\right] \\ &\equiv \Psi(t)\frac{f_1(t)}{\zeta(t)}. \end{aligned} \tag{4.2.36}$$

Here M_0 is a constant depending only on $\dot{f}_1$ and f_1.

Let $T_1 > 0$ be the greatest of all t such that $\Psi(t) \geqq 0$ for $\tau \leqq t$. We set

$$\tau_n^{**} = \min(\tau_n^*, T_1). \tag{4.2.37}$$

From $L_1 \overline{U}_n \leqq 0$, $L_1 u_{1,n} = 0$, the boundary conditions, the definition of $u_{j,n}$ and the maximum principle, it follows that for $t_n \leqq \tau \leqq \tau_n^{**}$

$$\begin{aligned} v_{1,n}(t) &\equiv \frac{\partial}{\partial x} u_{1,n}(y_n(t), t) \geqq \frac{\partial}{\partial x} \overline{U}_n(y_n(t), t) \\ &= -\frac{f_1(t)}{\sqrt{\pi t}} \frac{\exp(-y_n^2(t)/4t)}{\operatorname{erf}(y_n(t)/2\sqrt{t})}\left(1 - \operatorname{erf}\frac{y_n(t)}{2\sqrt{t}}\right). \end{aligned} \tag{4.2.38}$$

Sharpening this inequality by using (4.2.21), we find that

$$v_{1,n} \geqq -\frac{f_1(t)}{z_2(t)}\left(1 - \frac{z_2(t)}{\sqrt{\pi t}}\right)\left(1 - \frac{z_2^2(t)}{4t}\right) \quad \text{for } t_n \leqq t \leqq \tau_n^{**}. \tag{4.2.39}$$

Using the definition (4.1.4), we obtain

$$v_{1,n} \geqq \alpha_1 t^{\lambda_1}(1 - v_1^*(t)) \quad \text{for } t_n \leqq t \leqq \tau_n^{**}, \tag{4.2.40}$$

where $\lim_{t\to 0} v_1^* = 0$ and $v_1^*(t) > 0$ is not dependent on n.

Comparing (4.2.40), (4.2.31) and (4.2.2), we find that

$$\dot{y}_n(t) \geqq \alpha_1 t^{\lambda_1}(1 - v_1^*(t)) - \alpha_2 t^{\lambda_2}(1 + v_2^*(t)) \equiv \Psi^*(t). \tag{4.2.41}$$

Since

$$\lambda_1 \leqq \lambda_2; \quad \alpha_1 > \alpha_2 \text{ for } \lambda_1 = \lambda_2 \text{ and } \lim_{t\to 0} v_i^* = 0, \tag{4.2.42}$$

we see that there exists a number $T_2 > 0$, independent of n, such that

$$t^{-\lambda_1}\Psi^*(t) \geqq \beta > 0 \text{ for } 0 \leqq t \leqq T_2. \tag{4.2.43}$$

Obviously, for arbitrary n

$$\overline{T}^* = \min(T_1, T_2) \leqq \tau_n^{**}. \tag{4.2.44}$$

In fact, let us assume the opposite. Choose n such that $\tau_n^* < \overline{T}^*$. Let $\epsilon < \beta$. From the definition of τ_n^* it follows that there exists some $\tau < \tau_n^*$ such that

$$\tau^{-\lambda_1}\dot{y}_n(\tau) < \epsilon. \tag{4.2.45}$$

On the other hand, from the definition of T_1 and T_2 it follows that

$$\tau^{-\lambda_1}\dot{y}_n(\tau) \geqq \beta. \tag{4.2.46}$$

This contradiction proves our assertion.

Thus

$$\overline{T}^* = \min(T_1, T_2) \leqq \tau_n^{**}. \tag{4.2.47}$$

Since T_1 and T_2 are independent of n, and since

$$\tau_n^{**} = \min(\tau_n^*, T_1),$$

this shows that

$$T^* = \inf(\tau_n^*) \geqq \overline{T}^* > 0. \tag{4.2.48}$$

Hence from the definition of τ_n^* it follows that estimate (4.2.27) is valid on an interval not depending on n, i.e. that $\{v_{1,n}\}$ is uniformly bounded. Also from the fact that $\dot{y}_n(t) > 0$ on this interval and from the Stefan condition it follows that the sequences $\{v_{2,n}\}$ and $\{\dot{y}_n\}$ are uniformly bounded. From the uniform boundedness of the sequences $\{v_{i,n}(t)\}$ and $\{\dot{y}_n(t)\}$ it follows, as in the proof of the existence theorem in the large for problem B_1, that $\{v_{i,n}(t)\}$ and $\{\dot{y}_n(t)\}$ uniformly satisfy a Hölder condition of order $\lambda = 1/2$, which proves the equicontinuity of $\{v_{i,n}(t)\}$ $(i = 1, 2)$ and $\{\dot{y}_n(t)\}$, and therefore also proves Lemma 7.

We note further that Lemma 8 is clearly a consequence of estimates (4.2.21) and (4.2.28), which completes the proof of Theorem 14a.

REMARK. Clearly the sequence $\{y_n(t)\}$ approximates the boundary from below. If the initial conditions

$$u_{i,n}(x, t_n) = w_{i,1}(x, t_n) \quad (i = 1, 2); \qquad y_n(t_n) = z_1(t_n) \tag{4.2.49}$$

are given and we substitute the conditions

$$u_{i,n}(x, t_n) = w_{i,2}(x, t_n) \quad (i = 1, 2); \qquad y_n(t_n) = z_2(t_n), \tag{4.2.50}$$

then we may approximate the boundary from above. It is clear that here all the above estimates remain valid.

What is more, as initial conditions for defining $u_{i,n}$ and y_n we can use the conditions

$$u_{i,n}(x,t_n) = w_i(x,t_n) \quad (i=1,2); \qquad y_n(t_n) = z(t_n), \tag{4.2.51}$$

where

$$z(t) = \gamma t^{\nu+1}/(\nu+1) \tag{4.2.52}$$

and w_i is defined as a solution to the problem

$$\begin{aligned} &L_1 w_1 = 0, \quad 0 < x < z(t); \qquad L_2 w_2 = 0, \quad z(t) < x < 1; \\ &w_1|_{x=0} = f_1(t); \quad w_i|_{x=z(t)} = 0 \qquad (i=1,2); \\ &w_2|_{x=1} = f_2(t); \quad w_2|_{t=0} = \phi(x). \end{aligned} \tag{4.2.53}$$

It is possible to prove that here the whole argument employed in the proof of Theorem 14a remains in force. Hence the algorithm for problem A_1 described in §6, Chapter I, remains valid if on the interval $0 < t < t_0$ we have

$$v_i(t) = \alpha_i t^{\lambda_i} \quad (i=1,2); \qquad y(t) = \frac{\gamma t^{\nu+1}}{\nu+1}, \tag{4.2.54}$$

where γ, ν, α_1 and α_2 are defined in agreement with (4.1.3) and for $t > t_0$ the calculation is performed in agreement with the definition of this algorithm. Thus later on (see Supplement 3, §§4 and 5) we carry out this calculation in the case where the integral equation form of the problem of Stefan type is used.

We now prove

THEOREM 15. *In addition to* (4.1.1) *let the following conditions be satisfied*:

$$f_2(0) = \phi(1); \quad \dot{\phi}(x) \leqq 0, \ 0 < x < 1; \quad \dot{f}_2(t) \geqq 0, \ 0 < t < T_0. \tag{4.2.55}$$

Then the solution to problem A_1 *is unique.*

We note, first of all, that if u_1, u_2, y is any solution of problem A_1, then there exists some $T > 0$ such that simultaneously

$$\dot{f}_1(t) \leqq 0; \quad \dot{f}_2(t) \geqq 0; \quad \dot{y}(t) \geqq 0 \text{ for } 0 \leqq t \leqq T. \tag{4.2.56}$$

This follows from conditions (4.1.1), (4.2.55) and Theorem 11, according to which $\dot{y}(t)$ is nonnegative in some neighborhood of $t = 0$. Let D_i be the domain of definition of u_i, and Γ_i its parabolic boundary. Then

$\partial u_i/\partial x$ is continuous in D_i, parabolic in D_i and nonnegative on Γ_i $(i = 1, 2)$. This follows at once from (4.2.56), the nonnegativity of $\dot{\phi}(x)$, the integral representation $(2.2.6_i)$ $(i = 1, 2)$ and the definition (2.2.2). But in this case it follows from the maximum principle that

$$\frac{\partial u_i}{\partial x} \geqq 0; \quad (x, t) \in D_i \ (i = 1, 2). \tag{4.2.57}$$

Choose $\epsilon > 0$ arbitrarily small, and set

$$\begin{aligned} &z(t) = (1 - \epsilon)\, y(t); \quad \xi = x - y(t) + z(t) \equiv x - \epsilon y(t); \\ &\tau = t; \quad U_i(\xi, \tau) = u_i(x, t) \qquad (i = 1, 2). \end{aligned} \tag{4.2.58}$$

Then

$$\begin{aligned} &L_1 U_1 = -\epsilon \dot{y}(\tau) \frac{\partial U_1}{\partial \xi}; \quad -\epsilon y(\tau) < \xi < z(\tau), \quad \tau > 0; \\ &L_2 U_2 = -\epsilon \dot{y}(\tau) \frac{\partial U_2}{\partial \xi}; \quad z(\tau) < \xi < 1 - \epsilon y(\tau), \quad \tau > 0; \\ &U_1|_{\xi = -\epsilon y(\tau)} = f_1(\tau); \quad U_i|_{\xi = z(\tau)} = 0 \ (i = 1, 2); \\ &U_2|_{\tau = 0} = \phi(\xi); \quad U_2|_{\xi = 1 - \epsilon y(\tau)} = f_2(\tau). \end{aligned} \tag{4.2.59}$$

Since

$$\frac{\partial U_i}{\partial \xi} \equiv \frac{\partial u_i}{\partial x} \geqq 0; \quad \dot{y}(\tau) \geqq 0 \text{ for } 0 \leqq \tau \leqq T,$$

we see that $U_i(\xi, \tau)$ is superparabolic for $0 < \tau \leqq T$.

Now let $w_i(\xi, \tau)$ $(i = 1, 2)$ be defined by the conditions

$$\begin{aligned} &L_1 w_1 = 0, \ 0 < \xi < z(\tau); \quad L_2 w_2 = 0, \ z(\tau) < \xi < 1; \quad \tau > 0; \\ &w_1|_{\xi = 0} = f_1(\tau); \quad w_2|_{\xi = 1} = f_2(\tau); \\ &w_i|_{\xi = z(\tau)} = 0 \ (i = 1, 2); \quad w_2|_{\tau = 0} = \phi(\xi). \end{aligned} \tag{4.2.60}$$

Comparing U_1 and w_1 in the domain $0 < \xi < z(\tau)$, U_2 and w_2 in the domain $z(\tau) < \xi < 1 - \epsilon y(\tau)$ and using the continuity of u_i in $\overline{D}_i$ as well as the inequalities (4.2.55), (4.2.56), we find, by virtue of the superparabolicity of U_i, the parabolicity of w_i and the maximum principle, that

$$U_1|_{\xi = 0} \geqq w_1|_{\xi = 0} = f_1(\tau); \quad U_2|_{\xi = 1 - \epsilon y(\tau)} = f_2(\tau) \geqq w_2. \tag{4.2.61}$$

Hence from the vanishing of U_i and w_i for $\xi = z(\tau)$ it follows that

$$\left.\frac{\partial w_1}{\partial \xi}\right|_{\xi = z(\tau)} \geqq \left.\frac{\partial U_1}{\partial \xi}\right|_{\xi = z(\tau)}; \quad \left.\frac{\partial w_2}{\partial \xi}\right|_{\xi = z(\tau)} \leqq \left.\frac{\partial U_2}{\partial \xi}\right|_{\xi = z(\tau)}. \tag{4.2.62}$$

Replacing ξ by x and τ by t in the expression for w_i, and taking into account the equation

$$\frac{\partial U_i}{\partial \xi} \equiv \frac{\partial u_i}{\partial x} \quad (i = 1, 2),$$

we find, by virtue of the Stefan condition, that

$$\dot{z}(t) = (1 - \epsilon)\dot{y}(t) < \left(\frac{\partial u_1}{\partial x} - \frac{\partial u_2}{\partial x}\right)\Bigg|_{x=y(t)} \leqq \left(\frac{\partial w_1}{\partial x} - \frac{\partial w_2}{\partial x}\right)\Bigg|_{x=z(t)} \tag{4.2.63}$$

for $0 < t \leqq T$.

Now let u_1^*, u_2^*, y^* be a second solution of problem A_1. By virtue of Theorem 11, y and y^* touch at $t = 0$. Consequently for some time interval $0 < t \leqq t_0$ we have automatically

$$y^*(t) > z(t). \tag{4.2.64}$$

But this shows, by virtue of (4.2.63), that w_1, w_2, z and u_1^*, u_2^*, y satisfy every condition of Lemma 1. Consequently

$$z(t) = (1 - \epsilon)y(t) \leqq y^*(t) \text{ for } 0 \leqq t \leqq T. \tag{4.2.65}$$

Since here T is not dependent on ϵ and $\epsilon > 0$ is arbitrarily small, this shows that

$$y(t) \leqq y^*(t) \text{ for } 0 \leqq t \leqq T. \tag{4.2.66}$$

A reversal of the roles of the solutions u_1, u_2, y and u_1^*, u_2^*, y^* shows that

$$y(t) \geqq y^*(t). \tag{4.2.67}$$

Thus the coincidence of the boundaries y and y^* is proved. Since the solution to the first boundary problem of the heat equation in a domain with given regular boundary is uniquely determined, this shows that $u_i = u_i^*$ $(i = 1, 2)$, which completes the proof.

§3. The third boundary problem (A_3)

We turn now to an examination of problem A_3. We will assume that $f_i(t)$ $(i = 1, 2)$ is twice and $\phi(x)$ thrice continuously differentiable, and that conditions (3.2.6), (3.2.6$_1'$) and (3.2.6$_1''$) of Theorem 12 are fulfilled; in addition

$$\phi(1) = f_2(0); \qquad \dot{f}_1(t) \leqq 0;$$

$$\left|\frac{d}{dt}\ln f_1\right| < \frac{M}{t} \text{ for } 0 < t \leqq T_0; \quad (M = \text{const} > 0). \tag{4.3.1}$$

THEOREM 16. A. *There exist functions* $z(t)$ *and* $w_i(x,t)$ $(i=1,2)$ *such that for any sequence*

$$t_0 > t_1 > \cdots > t_n > \cdots, \quad \lim_{n\to\infty} t_n = 0, \tag{4.3.2}$$

the class of functions $\{u_{i,n}(x,t)\}$, $\{v_{i,n}(t)\}$ $(i=1,2)$, $\{y_n(t)\}$ *defined by the conditions*

$$\begin{aligned}
&L_1 u_{1,n} = 0, \quad 0 < x < y_n(t), \quad t > t_n;\\
&L_2 u_{2,n} = 0, \quad y_n(t) < x < 1, \quad t > t_n;\\
&\left(\frac{\partial}{\partial x} - h\right) u_{1,n}\Big|_{x=0} = -hf_1(t), \; u_{2,n}|_{x=1} = f_2(t), \quad t > t_n;\\
&u_{i,n}|_{t=0} = w_i(x,t_n), \; u_{i,n}|_{x=y_n(t)} = 0; \quad t > t_n \quad (i=1,2),\\
&\dot{y}_n(t) = v_{1,n}(t) - v_{2,n}(t), \quad t > t_n; \quad y_n(t_n) = z(t_n);\\
&v_{i,n}(t) = \frac{\partial}{\partial x} u_i\Big|_{x=y_n(t)},
\end{aligned} \tag{4.3.3}$$

is such that the sequence $\{v_{i,n}(t)\}$ $(i=1,2)$ *is uniformly bounded and equicontinuous.*

B. *For any sequence* $\{n_j\}$ *the uniform convergence of* $\{v_{i,n_j}(t)\}$ *implies the uniform convergence of the sequences* $\{u_{i,n_j}(x,t)\}$, $\{y_{n_j}(t)\}$ *and the convergence in problem* A_3 *to the solution* $u_i(x,t)$, $y(t)$ $(i=1,2)$.

C. *If in addition to the conditions of the theorem we add the conditions*

$$\begin{aligned}
&\phi(1) = f_2(0); \quad \dot{\phi}(x) \geqq 0, \qquad 0 \leqq x \leqq 1;\\
&f_2(t) \geqq 0, \quad 0 \leqq t \leqq T_0, \quad m_0 = 0,
\end{aligned} \tag{4.3.4}$$

then the solution to problem A_3 *is unique.*

REMARK. The demand C is made necessary only by the particular properties of our method of proof, and not by the essential nature of the problem. Thus, as in the case of problem A_1, the uniqueness of the solution follows from a theorem on the general solution of the problem without any regard to limitations on the data. In this connection the assertions B and A of the theorem could be replaced by the stronger assertion A*:

A*: *The sequences* $\{u_{i,n}\}$, $\{y_n\}$ *are uniformly convergent to the solutions of problem* A_1.

We will not present the proof of Theorem 16 in every detail, but limit

ourselves to an indication of those parts of the proof which are unlike those of problem A_3 or problem A_1.

Let

$$z_1 = hc_0 \frac{t^{m_0+1}}{m_0+1}(1-\delta\sqrt{t}); \quad z_2(t) = hc_0 \frac{t^{m_0+1}}{m_0+1}(1+\delta\sqrt{t}), \tag{4.3.5}$$

where m_0 and c_0 are defined by the conditions of Theorem 12, and $\delta = \text{const.} > 0$ will be chosen later. We further define $w_{2,j}$ by condition (4.1.5) and $w_{1,j}$ by the equations

$$\begin{aligned} w_{1,1} &= F_1(t)\left[1 - \frac{\operatorname{erf}(x/2\sqrt{t})}{\operatorname{erf}(z_1/2\sqrt{t})} - \frac{\operatorname{erf}(z_1/2\sqrt{t})}{2} + \frac{\operatorname{erf}^2(x/2\sqrt{t})}{2\operatorname{erf}(z_1/2\sqrt{t})}\right]; \\ w_{1,2} &= F_2(t)\left[1 - \frac{\operatorname{erf}(x/2\sqrt{t})}{\operatorname{erf}(z_2/2\sqrt{t})}\right], \end{aligned} \tag{4.3.6}$$

we choose $F_i(t)$ such that the boundary conditions

$$\left(\frac{\partial}{\partial x} - h\right) w_{1,j}\Big|_{x=0} = -hf_1(t) \quad (j = 1, 2) \tag{4.3.7}$$

are realized.

Introducing $w_{1,j}$ from (4.3.6) into (4.3.7), we find that

$$\begin{aligned} F_1(t) &= \frac{hf_1(t)\sqrt{\pi t}\operatorname{erf}\dfrac{z_1}{2\sqrt{t}}}{1 + h\sqrt{\pi t}\operatorname{erf}\dfrac{z_1}{2\sqrt{t}}\left(1 - \dfrac{1}{2}\operatorname{erf}\dfrac{z_1}{2\sqrt{t}}\right)}; \\ F_2(t) &= \frac{hf_1(t)\sqrt{\pi t}\operatorname{erf}\dfrac{z_2}{2\sqrt{t}}}{1 + h\sqrt{\pi t}\operatorname{erf}\dfrac{z_2}{2\sqrt{t}}}. \end{aligned} \tag{4.3.8}$$

It is easily seen that $\delta > 0$ can be chosen such that in some neighborhood of $t = 0$

$$\dot{z}_1(t) < v^*_{1,1}(t) - v^*_{2,1}(t); \quad \dot{z}_2(t) > v^*_{1,2}(t) - v^*_{2,2}(t). \tag{4.3.9}$$

Here

$$v^*_{i,j}(t) = \frac{\partial}{\partial x} w_{i,j}(z_j(t), t) \quad (i = 1, 2). \tag{4.3.9*}$$

In fact, for $v^*_{2,j}$ this follows by virtue of estimate (4.1.18). On the other hand, from (4.3.6) and (4.3.8) we have

$$v^*_{1,1}(t) = -\frac{hf_1(t)\exp\left(-\dfrac{z_1^2(t)}{4t}\right)\operatorname{erfc}\dfrac{z_1}{2\sqrt{t}}}{1+h\sqrt{\pi t}\operatorname{erf}\dfrac{z_1}{2\sqrt{t}}\left(1-\dfrac{1}{2}\operatorname{erf}\dfrac{z_1}{2\sqrt{t}}\right)};$$

(4.3.10)

$$v^*_{1,2}(t) = -\frac{hf_1(t)\exp\left(-\dfrac{z_2^2(t)}{4t}\right)}{1+h\sqrt{\pi t}\operatorname{erf}\left(\dfrac{z_2}{2\sqrt{t}}\right)}.$$

Estimating $v^*_{1,1}(t)$ from below and $v^*_{1,2}(t)$ from above, we find that for $z_1 < 2\sqrt{t}$

$$v^*_{1,1} > hc_0t^{m_0}\left[1-\frac{h^2c_0^2t^{2m_0+1}}{4(m_0+1)^2}\right]\left[1-\frac{hc_0}{\sqrt{\pi}}\frac{t^{m_0+1/2}}{m_0+1}\right]\frac{1-Mt}{1+h\sqrt{\pi t}};$$

(4.3.11)

$$v^*_{1,2} < hc_0t^{m_0}[1+Mt].$$

Here M is defined in agreement with (4.3.1).

Comparing (4.3.11), (4.3.5) and (4.1.18) shows that there exist $\delta > 0$ and $T_0 > 0$ such that the inequality (4.3.9) is valid for $0 < t \leqq T_0$. For fixed $\delta > 0$ we can find $T_1 \leqq T_0$ such that for $0 < t \leqq T_1$, $w_{1,1}$ is superparabolic and $w_{1,2}$ subparabolic everywhere in their domains of definition.

Let us assume now, as in the proof of Theorem 14, that

$$z(t) \equiv z_1(t); \quad w_i(x,t) \equiv w_{i,1}(x,t). \tag{4.3.12}$$

From definitions (4.3.5), (4.3.6) and (4.1.5), inequality (4.3.9), definition (4.3.4) and Lemma 4 it follows that

$$z_1(t) \leqq y_n(t) \leqq z_2(t) \text{ for } t_n \leqq t \leqq T, \tag{4.3.13}$$

where $T > t_n$ is independent of n. Using this result and Lemma 4 in place of Lemma 1, we can show, exactly as in §2, that there exists some $t^* > 0$ such that for every n

$$\dot{y}_n(t) \geqq 0 \text{ for } t_n \leqq t \leqq t^*. \tag{4.3.14}$$

For this purpose, as the comparison functions in (4.2.34) and (4.2.38) it is necessary to take the corresponding functions

$$\underline{u}_n(x,t) = F_{1,n}(t)\left(1 - \frac{x}{y_n(t)}\right);$$

$$\bar{u}_n(x,t) = F_{2,n}(t)\left[1 - \frac{\operatorname{erf}\frac{x}{2\sqrt{t}}}{\operatorname{erf}\frac{y_n(t)}{2\sqrt{t}}} - \frac{\operatorname{erf}\frac{y_n(t)}{2\sqrt{t}}}{2} + \frac{\operatorname{erf}^2\frac{x}{2\sqrt{t}}}{2\operatorname{erf}\frac{y_n(t)}{2\sqrt{t}}}\right], \tag{4.3.15}$$

where

$$F_{1,n} = \frac{hy_n(t)f_1(t)}{1 + hy_n(t)};$$

$$F_{2,n} = \frac{hf_1(t)\sqrt{\pi t}\operatorname{erf}\frac{y_n(t)}{2\sqrt{t}}}{1 + h\sqrt{\pi t}\operatorname{erf}\frac{y_n(t)}{2\sqrt{t}}\left(1 - \frac{1}{2}\operatorname{erf}\frac{y_n(t)}{2\sqrt{t}}\right)}. \tag{4.3.15*}$$

The remaining steps in the proof of assertions A and B of Theorem 16 are in no way different from those of Theorem 14, and therefore we will not elaborate upon them.

We now prove assertion C. Let u_1, u_2, y be some solution of problem A_3, and let

$$q_1 = \frac{\partial}{\partial x}u_1; \quad v_1 = q_1|_{x=y(t)-0};$$
$$p_1 = \frac{\partial}{\partial t}u_1 \equiv \frac{\partial^2}{\partial x^2}u_1; \quad r_1 = \left(\frac{\partial}{\partial x} - h\right)u_1. \tag{4.3.16}$$

We first show that from (2.5.3) (where we take $l = 0$ and $t_0 = 0$) it follows that

$$\lim_{t\to 0} u_1 = 0 \tag{4.3.17}$$

and therefore that

$$\lim_{t\to 0} q_1(0,t) = -hf_1(0) = hc_0 > 0. \tag{4.3.18}$$

At the same time, from (3.5.9), (3.5.10) and $(3.5.3_1)$ it follows that

$$\lim_{t\to 0} v_1 = hc_0 > 0. \tag{4.3.19}$$

Thus q_1 is continuous in the closure of the domain

(4.3.20) $$D_1 = \{0 < x < y(\tau);\ 0 < \tau \leqq t\},$$

is parabolic in D_1, and satisfies the conditions

(4.3.21) $$q_1|_{x=0} \geqq 0; \quad q_1|_{x=y(t)} \geqq 0$$

for $0 \leqq t \leqq T$, where $T > 0$ is sufficiently small. Hence from the maximum principle it follows that

(4.3.22) $$q_1 \geqq 0 \text{ for } (x,t) \in D_1; \quad 0 \leqq t \leqq T.$$

Moreover, from Remark 1 to Theorem 6 and from the validity of this theorem in the case of problem B_3 it follows that $v_1(t)$ is differentiable for $t > 0$. Hence it follows, in turn, that p_1 is continuous in $\overline{D}$, is parabolic in D and satisfies the boundary conditions

(4.3.23) $$\left(\frac{\partial}{\partial x} - h\right) p_1\Big|_{x=0} = -h\dot{f}_1(t); \quad p_1|_{x=y(t)} = -\dot{y}(t).$$

From the conditions of Theorem 16 and from Theorem 12 it follows that on some interval $0 \leqq t \leqq T$

(4.3.24) $$\left(\frac{\partial}{\partial x} - h\right) p_1\Big|_{x=0} \geqq 0; \quad p_1|_{x=y(t)} \leqq 0.$$

Consequently, by virtue of Lemma 3,

(4.3.25) $$p_1 \leqq 0 \text{ for } 0 \leqq x \leqq y(t); \quad 0 \leqq t \leqq T.$$

On the other hand,

(4.3.26) $$\frac{\partial}{\partial x} r_1 \equiv \frac{\partial}{\partial x} q_1 - h\frac{\partial}{\partial x} u_1 \equiv p_1 - hq_1.$$

Hence from (4.3.22) and (4.3.25) it follows that

(4.3.27) $$\frac{\partial}{\partial x} r_1 \leqq 0 \text{ for } 0 \leqq x \leqq y(t); \quad 0 \leqq t \leqq T.$$

Now choose $\epsilon > 0$ arbitrarily small. Setting

(4.3.28) $$\begin{aligned} &z(t) = (1-\epsilon)y(t); \quad \xi = x - y(t) + z(t); \quad \tau = t; \\ &U_i(\xi,\tau) = u_i(x,t) \quad (i = 1,2) \end{aligned}$$

as in §2, and using (4.3.22) and (4.3.27), we find that U_1 and U_2 are superparabolic in the domain $0 < \xi < z(\tau)$ and $z(\tau) < \xi < 1 - \epsilon y(\tau)$, for

$$\left(\frac{\partial}{\partial \xi} - h\right) U_1\Big|_{\xi=0} = \left(\frac{\partial}{\partial x} - h\right) u_1\Big|_{x=\epsilon y(\tau)}$$

(4.3.29)
$$\leqq \left(\frac{\partial}{\partial x} - h\right) u_1|_{x=0} = -hf_1(t);$$

$$U_2|_{\xi=1-\epsilon y(\tau)} = f_2(t).$$

We now define $w_i(\xi, \tau)$ by inequalities (4.2.60), but with the replacement of the condition $w_1|_{\xi=0} = f_1(\tau)$ by the condition

(4.3.30)
$$\left(\frac{\partial}{\partial \xi} - h\right) w_1\Big|_{\xi=0} = -hf_1(\tau);$$

and, using Lemma 4, we see, as in the proof of Theorem 15, that

(4.3.31)
$$\dot{z}(t) < \left(\frac{\partial}{\partial x} w_1 - \frac{\partial}{\partial x} w_2\right)\Big|_{x=z(t)} \quad \text{for } 0 < t \leqq T,$$

where $T > 0$ is not dependent on ϵ.

Consider a second solution u_1^*, u_2^*, y^* of problem A_3; if we use the fact that y and y^* coincide for $t = 0$ (by Theorem 12), and refer to Lemma 4 in place of Lemma 1, we see, as in §2, that

(4.3.32)
$$z(t) \leqq y^*(t) \quad \text{for } 0 \leqq t \leqq T,$$

for any value of $\epsilon > 0$.

The rest of the proof is exactly the same as for Theorem 15, and therefore we do not reproduce it.

CHAPTER V

THE DOUBLE-LAYER STEFAN PROBLEM WITH DEGENERATION OF ONE OF THE PHASES AT THE INITIAL MOMENT AND A DISCONTINUITY IN THE BOUNDARY AND INITIAL CONDITIONS

§1. Statement of the problem. Existence and uniqueness theorem [141]

We now return to the investigation of a problem of crystallization of a melt due to the immersion in it of a more refractory material; this problem was studied by approximation methods in Part 1, Chapter I, §3.

In dimensionless variables[1] the problem consists of determining $u(x,t)$ and $y(t)$ such that

$$a^2(x)\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < y(t); \quad t > 0, \ x \neq 1; \tag{5.1.1}$$

$$u(x,t) \text{ and } k(x)\frac{\partial u}{\partial x} \text{ are continuous for } x = 1, \ t > 0; \tag{5.1.2}$$

$$u|_{t=0} = 1, \ 0 < x < 1; \quad \left.\frac{\partial u}{\partial x}\right|_{x=0} = 0; \quad u|_{x=y(t)} = 0; \quad t > 0; \tag{5.1.3}$$

$$\dot{y}(t) = -\alpha + \beta v(t), \ t > 0; \quad y(0) = 1. \tag{5.1.4}$$

Here

$$v(t) = \left.\frac{\partial u}{\partial x}\right|_{x=y(t)-0}, \ t > 0; \tag{5.1.5}$$

$$a(x) = \begin{cases} a_1 = 1, \ 0 < x < 1 \\ a_2 = a = \text{const} > 0, \ 1 < x < y(t); \end{cases}$$
$$k(x) = \begin{cases} k_1 = \text{const} > 0, \ 0 < x < 1 \\ k_2 = \text{const} > 0, \ 1 < x < y(t) \end{cases} \tag{5.1.6}$$

and

$$\alpha = \text{const} > 0; \quad \beta = \text{const} < 0. \tag{5.1.7}$$

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[1] See §5.

Our goal is to prove the following theorem.

THEOREM 17. *There exists a unique solution to problem* (5.1.1)—(5.1.4) *such that* $u(x,t)$ *is uniformly bounded and*

$$v(t) \equiv \frac{w(t)}{\sqrt{t}} = \frac{w(0)}{\sqrt{t}} + v_1(t), \tag{5.1.8}$$

where $w(0) < 0$ *and* $v_1(t)$ *is continuous for* $t > 0$.

We will first prove the uniqueness of the solution under the assumption of its existence. This is a consequence of the following lemmas.

LEMMA 10. *The quantities* $w(0)$ *and* $v_1(0)$ *in* (5.1.8) *are uniquely defined when* $\beta v_1(0) - \alpha < 0$.

LEMMA 11. *Let* $u(x,t)$, $y(t)$ *be a solution of problem* (5.1.1)—(5.1.3) *for a given smooth boundary* $x = y(t)$ *of the form*

$$y(t) = 1 + 2\gamma a\sqrt{t} + O(t), \quad t > 0, \quad \gamma > 0 \tag{5.1.9}$$

and let Γ_η *be the arc of a circle* $(x-1)^2 + t^2 = \eta^2$ *belonging to the domain* $D = \{0 < x < y(t);\ t > 0\}$. *Further, let*

$$q(x,t) = \frac{\partial}{\partial x} u(x,t). \tag{5.1.10}$$

Then $q(x,t)$ *satisfies the conditions*

$$a^2(x)\frac{\partial^2 q}{\partial x^2} = \frac{\partial q}{\partial t}, \qquad 0 < x < y(t), \quad t > 0, \quad x \neq 1; \tag{5.1.11}$$

$$q|_{t=0} = 0,\ 0 < x < 1; \quad q|_{x=0} = 0; \quad q|_{x=y(t)} \leqq 0,\ t > 0,$$

$$k_1 q|_{x=1-0} = k_2 q|_{x=1+0},\ t > 0; \tag{5.1.12}$$

$$\left.\frac{\partial q}{\partial x}\right|_{x=1+0} = a^2 \left.\frac{\partial q}{\partial x}\right|_{x=1-0},\ t > 0, \tag{5.1.13}$$

where, moreover, there is some $\delta > 0$ *such that for every* $\eta \in (0, \delta)$

$$q < 0 \text{ for } (x,t) \in \Gamma_\eta. \tag{5.1.14}$$

We will prove the uniqueness theorem under the assumption of Lemmas 10 and 11. Choose $\epsilon > 0$ arbitrarily small and let

$$\begin{aligned} &\xi = 1 + (1-\epsilon)(x-1); \quad \tau = (1-\epsilon)^2 t; \\ &U(\xi,\tau) = u(x,t); \quad z(\tau) = 1 + (1-\epsilon)(y(t) - 1). \end{aligned} \tag{5.1.15}$$

Then $U(\xi,\tau)$, $z(\tau)$ is a solution to the problem obtained from (5.1.1)—(5.1.4) by replacing x and t by ξ and τ, $y(t)$ by $z(\tau)$ and $x=0$ by $\xi=\epsilon$. Condition (5.1.4) then results in the condition

$$\dot{z}(\tau) = -\frac{\alpha}{1-\epsilon} + \beta \frac{\partial}{\partial \xi} U\Big|_{\xi=z(\tau)-0}; \quad z(0)=1. \tag{5.1.16}$$

Let $V(\xi,\tau)$ be a solution of the problem (5.1.1)—(5.1.3) if in its formulation we write (ξ,τ) in place of (x,t) and $z(\tau)$ in place of $y(t)$. By definition $V(\xi,\tau)$ satisfies every condition of Lemma 11. Let

$$C_\eta = \{(\xi-1)^2+\tau^2<\eta^2\}; \quad D_\eta = D\setminus \overline{C}_\eta, \tag{5.1.17}$$

where

$$D = \{0<\xi<z(\tau); \quad 0<\tau; \quad z(\tau)>1\}. \tag{5.1.18}$$

By virtue of Lemma 11,

$$\partial V/\partial \xi = Q \leqq 0, \quad (\xi,\tau)\in \overline{\Gamma}_{\eta^*} = \overline{D}_\eta/D_\eta.$$

In addition, from the theorem of Friedman it follows that Q cannot attain its extreme value for $\xi = 1\pm 0$, since otherwise at an extreme point $\partial Q/\partial\xi$ would have equal left and right limits, contradicting condition (5.1.13).

Since $Q\leqq 0$ on $\overline{\Gamma}_{\eta^*}$ it follows from the maximum principle that

$$Q<0, \ (\xi,\tau)\in D_\eta. \tag{5.1.20}$$

Since $\eta>0$ is arbitrarily small and the arc Γ_η contracts to the point $\xi=1$, $\tau=0$ for $\eta\to 0$, it follows that

$$Q<0, \ (\xi,\tau)\in D. \tag{5.1.21}$$

In particular,

$$Q<0 \text{ for } \xi=\epsilon, \quad \tau>0. \tag{5.1.22}$$

Comparing $U(\xi,\tau)$ and $V(\xi,\tau)$ in the domain

$$D^\epsilon = \{\epsilon<\xi<z(\tau); \ \tau>0\},$$

shows, by virtue of the maximum principle and the theorem of Friedman, that

$$U(\xi,\tau)\leqq V(\xi,\tau), \ (\xi,\tau)\in D^\epsilon. \tag{5.1.23}$$

Since in addition

$$U|_{\xi=z(\tau)} = V|_{\xi=z(\tau)} = 0, \tag{5.1.24}$$

we see that

$$\left.\frac{\partial V}{\partial \xi}\right|_{\xi=z(\tau)} \leqq \left.\frac{\partial U}{\partial \xi}\right|_{\xi=z(\tau)}. \tag{5.1.25}$$

Thus from (5.1.16), since $\beta < 0$, we find that

$$\dot{z}(\tau) \leqq -\frac{\alpha}{1-\epsilon} + \beta \left.\frac{\partial V}{\partial \xi}\right|_{\xi=z(\tau)}, \quad \tau > 0. \tag{5.1.26}$$

Now let $u(\tilde{x}, t)$, $\tilde{y}(t)$ be a second solution of the original problem. We will write ξ and τ in place of x and t. By virtue of Lemma 10 we have

$$\begin{aligned}\lim_{\tau\to 0}[\dot{z}(\tau) - \dot{\tilde{y}}(\tau)] &= \lim_{\tau\to 0}\left\{\left[-\frac{\alpha}{1-\epsilon} + \beta\frac{w(0)}{\sqrt{\tau}} + \frac{\beta}{1-\epsilon} v_1\left(\frac{\tau}{(1-\epsilon)^2}\right)\right]\right. \\ &\qquad \left. - \left[-\alpha + \beta\frac{w(0)}{\sqrt{\tau}} + \beta\tilde{v}_1(\tau)\right]\right\} \\ &= \epsilon \cdot \frac{\beta v_1(0) - \alpha}{1-\epsilon} < 0.\end{aligned} \tag{5.1.27}$$

In addition,

$$z(0) = \tilde{y}(0) = 1. \tag{5.1.28}$$

Reasoning as in the proof of Lemma 1, we see that the inequality

$$z(\tau) \leqq \tilde{y}(\tau), \tag{5.1.29}$$

which is valid, by virtue of (5.1.27) and (5.1.28), on some sufficiently small interval $0 < \tau < \tau_0$, is valid for every $\tau > 0$ at which the solutions are both defined.

Returning to the variables x, t, we find that

$$\epsilon + (1-\epsilon) y\left(\frac{t}{(1-\epsilon)^2}\right) \leqq \tilde{y}(t) \tag{5.1.30}$$

for $0 \leqq t \leqq T$, where $T > 0$ is such that $u(x, \xi)$, $y(\tau)$ and $\tilde{u}(x, t)$, $\tilde{y}(t)$ are defined for every $t \in (0, T)$ and $\tau = t/(1-\epsilon)^2$.

In exactly the same way we prove that

$$-\epsilon + (1+\epsilon) y\left(\frac{t}{(1+\epsilon)^2}\right) \geqq \tilde{y}(t) \quad \text{for } 0 < t \leqq T^*, \tag{5.1.31}$$

where T^* is defined analogously to T. Since $\epsilon > 0$ is arbitrarily small, we see that

$$y(t) \equiv \tilde{y}(t). \tag{5.1.32}$$

Thus the boundary $x = y(t)$ is uniquely defined. Since by virtue of the theorem of Friedman and the maximum principle the solution of problem (5.1.1)—(5.1.3) for given $x = y(t)$ is uniquely defined, it follows that the assertion of uniqueness of the solution is valid under the assumption of validity of Lemmas 10 and 11.

§2. Reduction to an integral equation. Proof of Lemma 10

The validity of Lemmas 10 and 11 is established by an investigation of the integral equations to which problem (5.1.1)—(5.1.4) can be reduced.

Let $G(x, \xi, t)$ be the Green's function for the problem (5.1.1)—(5.1.3) on the halfline $x > 0$, i.e. a function satisfying the conditions

$$\begin{gathered} a^2(x)\frac{\partial^2}{\partial x^2}G = \frac{\partial}{\partial t}G,\ 0 < x < \infty,\ x \neq 1,\ t > 0; \\ \frac{\partial}{\partial x}G\Big|_{x=0} = 0; \quad G|_{t=0} = \delta(x-\xi),\ x \neq 1; \\ G|_{x=1-0} = G|_{x=1+0}; \quad k_1\frac{\partial}{\partial x}G\Big|_{x=1-0} = k_2\frac{\partial}{\partial x}G\Big|_{x=1+0}, \end{gathered} \tag{5.2.1}$$

where $\delta(x)$ is the Dirac δ-function. Considered as a function of ξ and t, G satisfies the conjugacy conditions[2)]

$$\begin{gathered} k_2 a_1^2 G|_{\xi=1-0} = k_1 a_2^2 G|_{\xi=1+0}; \\ a_1^2\frac{\partial}{\partial \xi}G\Big|_{\xi=1-0} = a_2^2\frac{\partial}{\partial \xi}G\Big|_{\xi=1+0}. \end{gathered} \tag{5.2.2}$$

Hence the solution $u(x, t)$ to the problem (5.1.1)—(5.1.4) must satisfy the condition[2)]

$$u(x,t) = \int_0^1 G(x,\xi,t)\,d\xi + a_2^2\int_0^t G(x, y(\tau), t-\tau)\,v(\tau)\,d\tau. \tag{5.2.3}$$

Let $g(x, \xi, t)$ be conjugate to $G(x, \xi, t)$, i.e. a solution to the equation

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[2)] See Supplement 2, §2.

(5.2.4) $$\frac{\partial}{\partial \xi} g = -\frac{\partial}{\partial x} G$$

such that

(5.2.4*) $$g|_{\xi=0} = 0,\ x > 0,\ t > 0.$$

Differentiating (5.2.3) with respect to x for $x \neq 0$, $x \neq 1$, $x \neq y(t)$, $t > 0$, using (5.2.4) and taking (5.2.4*) into account, we obtain

(5.2.5) $$\frac{\partial u}{\partial x} = -g(x, 1-0, t) + a_2^2 \int_0^t v(\tau) \frac{\partial}{\partial x} G(x, y(\tau), t-\tau)\, d\tau.$$

Passing to the limit for $x \to y(t) - 0$, we obtain, with the aid of the gap theorem for the heat potential of a double layer and (5.1.8),

(5.2.6) $$\begin{aligned} w(t) = &-2\sqrt{t} g(y(t), 1-0, t) \\ &+ 2a_2^2 \int_0^t w(\tau) \sqrt{\frac{t}{\tau}} \frac{\partial}{\partial x} G(y(t), y(\tau), t-\tau)\, d\tau. \end{aligned}$$

Adjoining condition (5.1.4), we write the integrated form

(5.2.7) $$y(t) = 1 - \alpha t + \beta \int_0^t \frac{w(\tau)}{\sqrt{\tau}}\, d\tau,$$

hence obtaining a complete system of integral equations of the problem.

We can express the functions $G(x, \xi, t)$ and $g(x, \xi, t)$ as in Supplement 2, §2. We note that

(5.2.8$_1$) $$g(x, 1-0, t) = \frac{\exp\left(-\frac{(x-1)^2}{4a^2 t}\right)}{(1+\lambda) a \sqrt{\pi t}} + O\left(\frac{\exp\left(-\frac{1}{t}\right)}{\sqrt{t}}\right),\ x > 1;$$

$$\frac{\partial}{\partial x} G(x, \xi, t)\Big|_{x>1;\xi>1}$$

(5.2.8$_2$) $$\begin{aligned} = -\frac{1}{4a^3 \sqrt{\pi} \cdot t^{3/2}} &\left\{ (x-\xi) \exp\left(-\frac{(x-\xi)^2}{4a^2 t}\right) \right. \\ &\left. - \delta(x+\xi-2) \exp\left(-\frac{(x+\xi-2)^2}{4a^2 t}\right) \right\} \\ &+ O\left(\frac{\exp\left(-\frac{1}{t}\right)}{t^{3/2}}\right). \end{aligned}$$

Here

$$\delta = \frac{1-\lambda}{1+\lambda}; \quad \lambda = \frac{k_2 a_1}{k_1 a_2} = \frac{k_2}{k_1 a}. \tag{5.2.9}$$

We now turn to the proof of Lemma 10. From (5.2.7) and (5.1.8) it follows that

$$\begin{aligned} y(t) &= 1 + 2\beta w(0)\sqrt{t} + \int_0^t (\beta v_1(\tau) - \alpha)\, d\tau \\ &\equiv 1 + 2\beta w(0)\sqrt{t} + O(t). \end{aligned} \tag{5.2.10}$$

Performing calculations analogous to §3 of Chapter III and using $(5.2.8_i)$ $(i = 1, 2)$, we find that

$$w(0) = -\frac{2\exp(-\gamma^2)}{(1+\lambda)a\sqrt{\pi}} - w(0)(I_1 - \delta I_2), \tag{5.2.11}$$

where

$$\gamma = \beta w(0)/a \tag{5.2.11*}$$

and

$$\begin{aligned} I_1 &= \frac{\gamma}{\sqrt{\pi}} \int_0^1 \frac{1-\sqrt{z}}{\sqrt{z}(1-z)^{3/2}} \exp\left(-\gamma^2 \frac{1-\sqrt{z}}{1+\sqrt{z}}\right) dz \equiv \operatorname{erf}\gamma; \\ I_2 &= \frac{\gamma}{\sqrt{\pi}} \int_0^1 \frac{1+\sqrt{z}}{\sqrt{z}(1-z)^{3/2}} \exp\left(-\gamma^2 \frac{1+\sqrt{z}}{1-\sqrt{z}}\right) dz = \operatorname{erfc}\gamma. \end{aligned} \tag{5.2.12}$$

We now rewrite (5.2.11) in the equivalent form

$$\gamma = -\frac{\beta \exp(-\gamma^2)}{\lambda a^2 \sqrt{\pi}} - \frac{\gamma}{\lambda} \operatorname{erf}\gamma \equiv f(\gamma) \tag{5.2.13}$$

and observe that

$$f(0) > 0; \quad \lim_{\gamma \to \infty} f(\gamma) = -\infty; \quad f'(\gamma) < 0 \text{ for } \gamma > 0, \tag{5.2.14}$$

We then see that the equation (5.2.11) has one and only one negative root $w(0)$. Thus $w(0) < 0$ is defined uniquely.

We next turn to the proof of the second part of Lemma 10. Let us rewrite (5.2.6) in the form

$$v(t) = -2g(y(t), 1-0, t) + 2a^2 \int_0^t v(\tau) \frac{\partial}{\partial x} G(y(t), y(\tau), t-\tau)\, d\tau. \tag{5.2.15}$$

We note that (5.2.11) can be rewritten in the equivalent form

$$\frac{w(0)}{\sqrt{t}} = -\frac{2\exp(-\gamma^2)}{(1+\lambda)a\sqrt{\pi t}} - \frac{w(0)\gamma}{\sqrt{\pi}}\int_0^t \frac{\sqrt{t}-\sqrt{\tau}}{\sqrt{\tau}(t-\tau)^{3/2}} \exp\left(-\gamma^2\frac{\sqrt{t}-\sqrt{\tau}}{\sqrt{t}+\sqrt{\tau}}\right) d\tau$$

(5.2.16)
$$+\frac{\delta w(0)\gamma}{\sqrt{\pi}}\int_0^t \frac{\sqrt{t}+\sqrt{\tau}}{\sqrt{\tau}(t-\tau)^{3/2}} \exp\left(-\gamma^2\frac{\sqrt{t}+\sqrt{\tau}}{\sqrt{t}-\sqrt{\tau}}\right) d\tau.$$

But by definition

(5.2.17)
$$v_1(t) = v(t) - w(0)/\sqrt{t}.$$

Consequently

(5.2.18)
$$v_1(t) = \sum_{j=0}^{2} J_j(t),$$

where

(5.2.19)
$$J_0 = -2g(y(t), 1-0, t) + \frac{2\exp(-\gamma^2)}{(1+\lambda)a\sqrt{\pi t}};$$

(5.2.20)
$$J_1 = 2a^2\int_0^t v_1(\tau)\frac{\partial}{\partial x}G(y(t), y(\tau), t-\tau)\,d\tau;$$

$$J_2 = \int_0^t \frac{w(0)}{\sqrt{\tau}}\left\{2a^2\frac{\partial}{\partial x}G(y(t), y(\tau), t-\tau)\right.$$

$$+\frac{\gamma}{\sqrt{\pi}}\frac{1}{(t-\tau)^{3/2}}\left[(\sqrt{t}-\sqrt{\tau})\exp\left(-\gamma^2\frac{\sqrt{t}-\sqrt{\tau}}{\sqrt{t}+\sqrt{\tau}}\right)\right. \quad (5.2.21)$$

$$\left.\left.-\delta(\sqrt{t}+\sqrt{\tau})\exp\left(-\gamma^2\frac{\sqrt{t}+\sqrt{\tau}}{\sqrt{t}-\sqrt{\tau}}\right)\right]\right\}d\tau.$$

Using (5.2.8$_i$) $(i = 1, 2)$ and omitting the terms tending to zero for $t \to 0$ faster than any power of t, we obtain

(5.2.22)
$$J_0 = -\frac{2}{(1+\lambda)a\sqrt{\pi}}\cdot\frac{\exp\left(-\frac{(y(t)-1)^2}{4a^2t}\right) - \exp(-\gamma^2)}{\sqrt{t}};$$

$$J_1 = -\frac{1}{2a\sqrt{\pi}}\int_0^t v_1(\tau)\left\{\frac{y(t)-y(\tau)}{(t-\tau)^{3/2}}\exp\left[-\frac{(y(t)-y(\tau))^2}{4a^2(t-\tau)}\right]\right.$$

(5.2.23)
$$\left.-\delta\frac{y(t)+y(\tau)-2}{(t-\tau)^{3/2}}\exp\left[-\frac{(y(t)+y(\tau)-2)^2}{4a^2(t-\tau)}\right]\right\}d\tau;$$

(5.2.24)
$$J_2 = J_2' + J_2''.$$

Here

$$J_2' = -\frac{w(0)}{\sqrt{\pi}}\int_0^t \Bigg[\frac{y(t)-y(\tau)}{2a\sqrt{\tau}(t-\tau)^{3/2}}\exp\left(\frac{(y(t)-y(\tau))^2}{4a^2(t-\tau)}\right) - \frac{\gamma}{\sqrt{\tau}}\frac{\sqrt{t}-\sqrt{\tau}}{(t-\tau)^{3/2}}\exp\left(-\gamma^2\frac{\sqrt{t}-\sqrt{\tau}}{\sqrt{t}+\sqrt{\tau}}\right)\Bigg]d\tau; \tag{5.2.25}$$

$$J_2'' = \frac{\delta w(0)}{\sqrt{\pi}}\int_0^t \Bigg[\frac{y(t)+y(\tau)-2}{2a\sqrt{\tau}(t-\tau)^{3/2}}\exp\left(-\frac{(y(t)+y(\tau)-2)^2}{4a^2(t-\tau)}\right) - \frac{\gamma}{\sqrt{\tau}}\frac{\sqrt{t}+\sqrt{\tau}}{(t-\tau)^{3/2}}\exp\left(-\gamma^2\frac{\sqrt{t}+\sqrt{\tau}}{\sqrt{t}-\sqrt{\tau}}\right)\Bigg]d\tau. \tag{5.2.26}$$

Taking the limit in (5.2.22) for $t\to 0$ and using (5.2.10), we find that

$$\lim_{t\to 0} J_0 = \frac{2\gamma\exp(-\gamma^2)}{a^2(1+\lambda)\sqrt{\pi}}[\beta v_1(0)-\alpha] = [\beta v_1(0)-\alpha]I_1(\gamma). \tag{5.2.27}$$

Substituting $\tau = tz$ in the right-hand side of (5.2.23) and taking the limit for $t\to 0$, we find as above that

$$\lim_{t\to 0} J_1 = -\frac{v_1(0)\gamma}{\sqrt{\pi}}\int_0^1 \frac{1}{(1-z)^{3/2}}\Bigg[(1-\sqrt{z})\exp\left(-\gamma^2\frac{1-\sqrt{z}}{1+\sqrt{z}}\right) - \delta(1+\sqrt{z})\exp\left(-\gamma^2\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\Bigg]dz \tag{5.2.28}$$

$$= -v_1(0)\,I_0(\gamma).$$

Furthermore,

$$J_2' = j_2' + j_2'' = -\frac{w(0)}{2a\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{\tau}}\frac{y(t)-y(\tau)-2a\gamma(\sqrt{t}-\sqrt{\tau})}{(t-\tau)^{3/2}} \cdot \exp\left(-\frac{(y(t)-y(\tau))^2}{4a^2(t-\tau)}\right)d\tau - \frac{w(0)\gamma}{\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{\tau}}\frac{\sqrt{t}-\sqrt{\tau}}{(t-\tau)^{3/2}} \cdot \left[\exp\left(-\frac{(y(t)-y(\tau))^2}{4a^2(t-\tau)}\right) - \exp\left(-\gamma^2\frac{\sqrt{t}-\sqrt{\tau}}{\sqrt{t}+\sqrt{\tau}}\right)\right]d\tau. \tag{5.2.29}$$

Setting $\tau = tz$ and taking the limit under the integral sign, we obtain

$$\lim_{t\to 0} j_2' = -\frac{w(0)}{2a\sqrt{\pi}}\int_0^1 \frac{\beta v_1(0)-\alpha}{(1-z)^{1/2}}\exp\left(-\gamma^2\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)\frac{dz}{\sqrt{z}} = (\beta v_1(0)-\alpha)\,I_2(\gamma); \tag{5.2.30$_1$}$$

$$\lim_{t\to 0} j_2'' = \frac{w(0)\gamma^2}{a\sqrt{\pi}} \int_0^1 \frac{\beta v_1(0) - \alpha}{\sqrt{z}\sqrt{1-z}} \frac{1-\sqrt{z}}{1+\sqrt{z}} \exp\left(-\gamma^2 \frac{1-\sqrt{z}}{1+\sqrt{z}}\right) dz$$

$$= (\beta v_1(0) - \alpha) I_3(\gamma). \tag{5.2.30$_2$}$$

In the same way

$$\lim_{t\to 0} J_2'' = j_2^* + j_2^{**}, \tag{5.2.31}$$

where

$$j_2^* = \frac{\delta w(0)}{2a\sqrt{\pi}} \int_0^1 \frac{\beta v_1(0) - \alpha}{(1-z)^{3/2}} \frac{(1+z)}{\sqrt{z}} \exp\left(-\gamma^2 \frac{1+\sqrt{z}}{1-\sqrt{z}}\right) dz$$

$$= (\beta v_1(0) - \alpha) I_4(\gamma); \tag{5.2.32$_1$}$$

$$j_2^{**} = -\frac{\delta\gamma^2 w(0)}{a\sqrt{\pi}} \int_0^1 \frac{\beta v_1(0) - \alpha}{(1-z)^{3/2}} \cdot \frac{1+\sqrt{z}}{1-\sqrt{z}} \cdot \frac{1+z}{\sqrt{z}} \exp\left(-\gamma^2 \frac{1+\sqrt{z}}{1-\sqrt{z}}\right) dz$$

$$= (\beta v_1(0) - \alpha) I_5(\gamma). \tag{5.2.32$_2$}$$

Combining the resulting limiting relations, we obtain

$$v_1(0)(1 + I_0(\gamma)) = (\beta v_1(0) - \alpha) \sum_{j=1}^{5} I_j(\gamma), \tag{5.2.33}$$

which can be rewritten in the form

$$\beta v_1(0) - \alpha = -\frac{\alpha(1 + I_0(\gamma))}{1 + I_0(\gamma) - \beta \sum_{j=1}^{5} I_j(\gamma)}, \tag{5.2.34}$$

if only

$$\tilde{I} = 1 + I_0 - \beta \sum_{j=1}^{5} I_j \neq 0. \tag{5.2.35}$$

Thus to complete the proof of Lemma 10 it is sufficient to show that

$$\tilde{I} > 0; \quad 1 + I_0 > 0. \tag{5.2.36}$$

Using (5.2.11)—(5.2.13), we rewrite (5.2.27)—(5.2.32$_2$) in the form

$$I_0(\gamma) = \frac{\gamma}{\sqrt{\pi}} \int_0^1 \frac{1-\sqrt{z}}{(1-z)^{3/2}} \exp\left(-\gamma^2 \frac{1-\sqrt{z}}{1+\sqrt{z}}\right) dz$$

$$- \frac{\delta\gamma}{\sqrt{\pi}} \int_0^1 \frac{1+\sqrt{z}}{(1-z)^{3/2}} \exp\left(-\gamma^2 \frac{1+\sqrt{z}}{1-\sqrt{z}}\right) dz; \tag{5.2.37$_1$}$$

$$(5.2.37_2)\qquad -\beta I_1(\gamma) = -\frac{2\exp(-\gamma^2)}{(1+\lambda)a\sqrt{\pi}}\cdot\frac{\gamma^2}{w(0)} = \gamma^2(1+\operatorname{erf}\gamma-\delta\operatorname{erfc}\gamma);$$

$$(5.2.37_3)\qquad -\beta(I_2(\gamma)+I_3(\gamma)) = \frac{\gamma}{\sqrt{\pi}}\int_0^1 \frac{1}{\sqrt{z}\sqrt{1-z}}\left(\frac{1}{2}-\gamma^2\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)\cdot\exp\left(-\gamma^2\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)dz;$$

$$(5.2.37_4)\qquad -\beta(I_4(\gamma)+I_5(\gamma)) = -\frac{\delta\gamma}{\sqrt{\pi}}\int_0^1 \frac{1+\sqrt{z}}{(1-z)^{3/2}}\cdot\frac{1+z}{1+\sqrt{z}}\cdot\left(\frac{1}{2}-\gamma^2\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\exp\left(-\gamma^2\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\frac{dz}{\sqrt{z}}.$$

Again using the substitution

$$\gamma\sqrt{\frac{1-\sqrt{z}}{1+\sqrt{z}}} = x; \quad \gamma\sqrt{\frac{1+\sqrt{z}}{1-\sqrt{z}}} = \xi,$$

we reduce the expression $\widetilde{I}$ to the form

$$(5.2.38)\qquad \begin{aligned}\widetilde{I}(\gamma) &= 1+\gamma^2(1+\operatorname{erf}\gamma-\delta\operatorname{erfc}\gamma)\\ &+\frac{2}{\sqrt{\pi}}\int_0^\gamma \frac{\exp(-x^2)}{\gamma^2+x^2}\left[\gamma^2-x^2+2\gamma^2\left(\frac{1}{2}-x^2\right)\right]dx\\ &-\frac{2\delta}{\sqrt{\pi}}\int_\gamma^\infty \frac{\exp(-\xi^2)}{\gamma^2+\xi^2}\left[\xi^2-\gamma^2+\frac{\xi^4+\gamma^4}{\xi^2}\left(\frac{1}{2}-\xi^2\right)\right]d\xi,\end{aligned}$$

or (as is obvious), in the same way, to the form

$$(5.2.39)\qquad \widetilde{I}(\gamma) = 1-\operatorname{erf}\gamma+\gamma^2(1+\operatorname{erf}\gamma)+I^*(\gamma)-\delta I^{**}(\gamma),$$

where

$$(5.2.40)\qquad I^*(\gamma) = \frac{4\gamma^2}{\sqrt{\pi}}\int_0^\gamma \frac{\exp(-x^2)}{x^2+\gamma^2}\left(\frac{3}{2}-x^2\right)dx;$$

$$(5.2.41)\qquad \begin{aligned}I^{**}(\gamma) &= \frac{2}{\sqrt{\pi}}\int_\gamma^\infty \exp(-\xi^2)\left\{\gamma^2+\frac{\xi^2-\gamma^2}{\xi^2+\gamma^2}+\frac{\xi^4+\gamma^4}{(\xi^2+\gamma^2)\xi^2}\left(\frac{1}{2}-\xi^2\right)\right\}d\xi\\ &\equiv \frac{2}{\sqrt{\pi}}\int_\gamma^\infty \exp(-\xi^2)\left(\frac{3}{2}-\xi^2\right)d\xi-\frac{4\gamma^2}{\sqrt{\pi}}\int_\gamma^\infty \exp(-\xi^2)\frac{3/2-\xi^2}{\xi^2+\gamma^2}d\xi\\ &+\frac{\gamma^2}{\sqrt{\pi}}\int_\gamma^\infty \frac{\exp(-\xi^2)}{\xi^2}d\xi = j_1+j_2+j_3.\end{aligned}$$

It is easily seen that

$$j_1 = \operatorname{erfc}\gamma - \frac{\gamma \exp(-\gamma^2)}{\sqrt{\pi}}; \quad j_3 = \frac{\gamma}{\sqrt{\pi}} \exp(-\gamma^2) - \gamma^2 \operatorname{erfc}\gamma. \tag{5.2.42}$$

Further, by virtue of the theorem of the mean, we have

$$j_2 = -\frac{3\gamma^2}{x_0^2+\gamma^2} \operatorname{erfc}\gamma + \frac{2x_1^2\gamma^2}{x_1^2+\gamma^2} \operatorname{erfc}\gamma; \quad x_0 > \gamma,\ x_1 > \gamma, \tag{5.2.43}$$

which can also be written in the form

$$J_2 = \left(-\frac{3}{2}\theta + 2\theta_1\gamma^2\right) \operatorname{erfc}\gamma, \quad 0 < \theta,\ \theta_1 < 1. \tag{5.2.44}$$

Combining (5.2.42)—(5.2.44), we obtain

$$I^{**}(\gamma) = \operatorname{erfc}\gamma \cdot \left(1 - \frac{3}{2}\theta\right) - \gamma^2(1 - 2\theta_1)\operatorname{erfc}\gamma. \tag{5.2.45}$$

Further,

$$I^*(\gamma) = \frac{4\gamma^2}{\sqrt{\pi}} \int_0^\gamma \exp(-x^2)\phi(x^2)\,dx; \quad \phi(x) = \frac{3/2 - x}{x+\gamma^2}. \tag{5.2.46}$$

Since

$$\frac{d\phi}{dx} < 0; \quad x > 0,$$

we see that

$$I^*(\gamma) > \frac{4\gamma^2}{\sqrt{\pi}} \frac{3/2 - \gamma^2}{2\gamma^2} \int_0^\gamma \exp(-x^2)\,dx = \left(\frac{3}{2} - \gamma^2\right) \operatorname{erf}\gamma. \tag{5.2.47}$$

Introducing (5.2.45) and (5.2.47) into (5.2.39), we obtain

$$\begin{aligned} \tilde{I}(\gamma) > 1 + \frac{1}{2}\operatorname{erf}\gamma - \delta\left(1 - \frac{3}{2}\theta\right)\operatorname{erfc}\gamma \\ + \gamma^2[1 + \delta(1 - 2\theta_1)\operatorname{erfc}\gamma]. \end{aligned} \tag{5.2.48}$$

But $-1 < \delta(1 - 2\theta_1) < 1$ and $-1 < \delta(-3\theta/2) < 1$, since $|\delta| < 1$ and $0 < \theta,\ \theta_1 < 1$. Consequently

$$\begin{aligned} \tilde{I}(\gamma) &> 1 + \frac{1}{2}\operatorname{erf}\gamma - \operatorname{erfc}\gamma + \gamma^2(1 - \operatorname{erfc}\gamma) \\ &= \left(\frac{3}{2} + \gamma^2\right)\operatorname{erf}\gamma > 0. \end{aligned} \tag{5.2.49}$$

Analogously, by virtue of $(5.2.37_1)$, (5.2.12) and the theorem of the mean,

$$1 + I_0(\gamma) = 1 + \Theta \operatorname{erf}\gamma - \delta\Theta_1 \operatorname{erfc}\gamma; \quad 0 < \Theta,\ \Theta_1 < 1. \tag{5.2.50}$$

Consequently

$$1 + I_0(\gamma) > 0. \tag{5.2.51}$$

Thus we have proved inequality (5.2.36) and, together with it, Lemma 10.

§3. Proof of Lemma 11

We turn to the proof of Lemma 11. First we note that (5.2.5) implies the existence of the limit of $\partial^2 u/\partial x^2$ for $x \to 1 \pm 0$, if $t > 0$ and $y(t) > 1$. Hence by the conditions of the problem (5.1.1)—(5.1.4) it obviously follows that q satisfies every condition of (5.1.11)—(5.1.13). Thus it is only necessary to prove the validity of the inequality

$$q < 0 \text{ for } (x,t) \in \Gamma_\eta. \tag{5.3.1}$$

We examine the curve

$$x(t) = 1 + c(y(t) - 1), \quad 0 < c < 1 \tag{5.3.2*}$$

and find that

$$j(c) = \lim_{t \to 0} \sqrt{t} \frac{\partial}{\partial x} u(x(t), t). \tag{5.3.2}$$

From (5.2.5) it follows that

$$\begin{aligned} \sqrt{t} \frac{\partial}{\partial x} u(x(t),t) &= -\sqrt{t} g(x(t), 1-0, t) \\ &\quad + a^2 \int_0^t w(\tau) \sqrt{\frac{t}{\tau}} \frac{\partial}{\partial x} G(x(t), y(\tau), t-\tau)\, d\tau. \end{aligned} \tag{5.3.3}$$

Hence from $(5.2.8_i)$ $(i = 1, 2)$ we have

$$\begin{aligned} j(c) = &-\frac{\exp(-c^2\gamma^2)}{(1+\lambda)a\sqrt{\pi}} \\ &- \frac{w(0)}{4a\sqrt{\pi}} \lim_{t \to 0} \int_0^t \left\{ \sqrt{\frac{t}{\tau}} \frac{x(t) - y(\tau)}{(t-\tau)^{3/2}} \cdot \exp\left(-\frac{(x(t) - y(\tau))^2}{4a^2(t-\tau)} \right) \right. \\ &\quad \left. - \delta \sqrt{\frac{t}{\tau}} \frac{x(t) + y(\tau) - 2}{(t-\tau)^{3/2}} \right. \\ &\quad \left. \cdot \exp\left(-\frac{(x(t) + y(\tau) - 2)^2}{4a^2(t-\tau)} \right) \right\} d\tau. \end{aligned} \tag{5.3.4}$$

Repeating the derivation of (5.2.11), we obtain

$$j(c) = -\frac{\exp(-c^2\gamma^2)}{(1+\lambda)a\sqrt{\pi}} - \frac{w(0)}{2}\,(I_1(c) - \delta I_2(c)), \tag{5.3.5}$$

where

$$I_1(c) = \frac{\gamma}{\sqrt{\pi}}\int_0^1 \frac{c-\sqrt{z}}{\sqrt{z}(1-z)^{3/2}} \exp\left(-\gamma^2\frac{(c-\sqrt{z})^2}{1-z}\right) dz;$$
$$I_2(c) = \frac{\gamma}{\sqrt{\pi}}\int_0^1 \frac{c+\sqrt{z}}{\sqrt{z}(1-z)^{3/2}} \exp\left(-\gamma^2\frac{(c+\sqrt{z})^2}{1-z}\right) dz. \tag{5.3.6}$$

Setting

$$\gamma\frac{c+\sqrt{z_3}}{\sqrt{1-z_3}} = \xi; \quad \gamma\frac{c-\sqrt{z_1}}{\sqrt{1-z_1}} = x \text{ for } 0 < z_1 < c^2;$$
$$\gamma\frac{c-\sqrt{z_2}}{\sqrt{1-z_2}} = -x \text{ for } c^2 < z_2 < 1, \tag{5.3.7}$$

we obtain

$$I_1(c) = I_1'(c) + I_1''(c) = \frac{2}{\sqrt{\pi}}\int_0^{c\gamma} \frac{c-\sqrt{z_1(x)}}{1-c\sqrt{z_1(x)}} \exp(-x^2)\,dx$$
$$+\frac{2}{\sqrt{\pi}}\int_0^{\infty} \frac{c-\sqrt{z_2(x)}}{1-c\sqrt{z_2(x)}} \exp(-x^2)\,dx; \tag{5.3.8$_1$}$$

$$I_2(c) = \frac{2}{\sqrt{\pi}}\int_{c\gamma}^{\infty} \frac{c+\sqrt{z_3(\xi)}}{1+c\sqrt{z_3(\xi)}} \exp(-\xi^2)\,d\xi, \tag{5.3.8$_2$}$$

where

$$\sqrt{z_1(x)} = \frac{c\gamma^2}{x^2+\gamma^2} - \sqrt{\frac{c^2\gamma^4}{(x^2+\gamma^2)^2} + \frac{x^2-c^2\gamma^2}{x^2+\gamma^2}};$$
$$\sqrt{z_2(x)} = \frac{c\gamma^2}{x^2+\gamma^2} + \sqrt{\frac{c^2\gamma^4}{(x^2+\gamma^2)^2} + \frac{x^2-c^2\gamma^2}{x^2+\gamma^2}}; \tag{5.3.9}$$
$$\sqrt{z_3(x)} = -\frac{c\gamma^2}{x^2+\gamma^2} + \sqrt{\frac{c^2\gamma^4}{(x^2+\gamma^2)^2} + \frac{x^2-c^2\gamma^2}{x^2+\gamma^2}}.$$

From (5.3.8$_1$) and (5.3.8$_2$) it follows that

$$-I_1(c) + \delta I_2(c) = j_1(c) + j_2(c), \tag{5.3.10}$$

where

$$j_1(c) = -\frac{2}{\sqrt{\pi}} \int_0^{c\gamma} \left(\frac{c - \sqrt{z_1}}{1 - c\sqrt{z_1}} + \frac{c - \sqrt{z_2}}{1 - c\sqrt{z_1}} \right) \exp(-x^2)\, dx;$$

(5.3.11)

$$j_2(c) = \frac{2}{\sqrt{\pi}} \int_{c\gamma}^{\infty} \left(\frac{\sqrt{z_2} - c}{1 - c\sqrt{z_2}} + \delta \frac{c + \sqrt{z_3}}{1 + c\sqrt{z_3}} \right) \exp(-x^2)\, dx.$$

A calculation gives

$$\frac{c - \sqrt{z_1}}{1 - c\sqrt{z_1}} + \frac{c - \sqrt{z_2}}{1 - c\sqrt{z_2}} \equiv 0;$$

(5.3.12)

$$\frac{\sqrt{z_2} - c}{1 - c\sqrt{z_2}} + \delta \frac{c + \sqrt{z_3}}{1 + c\sqrt{z_3}} \equiv \frac{(1 + \delta)x}{\sqrt{x^2 - c^2\gamma^2 + \gamma^2}}.$$

Consequently $j_1 \equiv 0$ and

$$-I_1(c) + \delta I_2(c) = j_2(c) = \frac{2}{\sqrt{\pi}} \int_{c\gamma}^{\infty} \frac{(1 + \delta)x \exp(-x^2)}{\sqrt{x^2 - c^2\gamma^2 + \gamma^2}}\, dx. \tag{5.3.13}$$

Substituting $\tau = \sqrt{x^2 - c^2\gamma^2 + \gamma^2}$ gives

$$-I_1(c) + \delta I_2(c) = (1 + \delta) \exp[\gamma^2(1 - c^2)] \operatorname{erfc} \gamma. \tag{5.3.14}$$

Inserting (5.3.14) into (5.3.5) gives

$$j(c) = -\frac{\exp(-c^2\gamma^2)}{(1 + \lambda)a\sqrt{\pi}} + \frac{1 + \delta}{2} \exp[\gamma^2(1 - c^2)] \operatorname{erfc} \gamma \cdot w(0). \tag{5.3.15}$$

Since $w(0) < 0$, we find from (5.3.2) that

$$\lim_{t \to 0} \sqrt{t} \frac{\partial}{\partial x} u(x(t), t) < 0 \tag{5.3.16}$$

if $x(t) = 1 + c(y(t) - 1)$, $0 < c < 1$.

We note that in the limit for $c \to +0$

$$j(0) = -\frac{1}{(1 + \lambda)a\sqrt{\pi}} + \frac{1 + \delta}{2} \exp(\gamma^2) \operatorname{erfc} \gamma \cdot w(0) < 0. \tag{5.3.17}$$

Further

$$j(1 - 0) = -\frac{\exp(-\gamma^2)}{(1 + \lambda)a\sqrt{\pi}} + \frac{1 + \delta}{2} \operatorname{erfc} \gamma \cdot w(0). \tag{5.3.18}$$

Using (5.2.11) shows that

$$j(1-0) = w(0). \tag{5.3.19}$$

Hence by (5.3.17) we see that (5.3.16) is valid on the interval $0 \leqq c \leqq 1$. Thus it is obvious that the passage to the limit for $t \to 0$ and $1 \leqq x \leqq y(t)$ in the expression $\sqrt{t}\,q(x,t)$ is realized uniformly. But this shows that there exists some $\delta > 0$ such that in the part Γ_η of the arc of the circle $\Gamma_\eta = \{(x-1)^2 + t^2 = \eta^2\}$ belonging to the domain $1 < x < y(t)$, $t > 0$ the inequality (5.1.14) remains valid. In particular,

$$q(1+0,t) < 0 \text{ for } 0 < t < \delta. \tag{5.3.20}$$

Since $q(1-0,t) = a^2 q(1+0,t)$, we conclude that

$$q(1-0,t) < 0 \text{ for } 0 < t < \delta. \tag{5.3.21}$$

In order to prove the validity of inequality (5.1.14) in the domain $0 < x < 1$, $t > 0$, we note that for $x < 1$, $u(x,t)$ can be expressed in the form

$$\begin{aligned} u(x,t) = \int_0^1 G_2(x,\xi,t)\,d\xi + \int_0^t u(0,\tau)\frac{\partial}{\partial \xi} G_2(x,0,t-\tau)\,d\tau \\ + \int_0^t q(1-0,\tau)\,G_2(x,1,t-\tau)\,d\tau, \end{aligned} \tag{5.3.22}$$

where G_i is the Green's function for the ith boundary problem with the equation $L_1 u = 0$ on the halfline $x < 1$ (see (0.2.8) and (0.2.9)). Differentiating (5.3.22) with respect to x for $0 < x < 1$, $t > 0$ and using the conjugacy of G_2 and G_1, we find that

$$\begin{aligned} q(x,t) = G_1(x,0,t) + \int_0^t u(0,\tau)\frac{\partial}{\partial \tau} G_1(x,0,t-\tau)\,d\tau \\ + \int_0^t q(1-0,\tau)\frac{\partial}{\partial x} G_2(x,1,t-\tau)\,d\tau. \end{aligned} \tag{5.3.23}$$

We have

$$\frac{\partial}{\partial x} G_2(x,1,t) = \frac{1-x}{2\sqrt{\pi}t^{3/2}} \exp\left(-\frac{(1-x)^2}{4t}\right);$$

$$\begin{aligned} \frac{\partial}{\partial t} G_1(x,0,t) = -\frac{1}{4\sqrt{\pi}t^{3/2}}\left[\left(1-\frac{x^2}{2t}\right)\exp\left(-\frac{x^2}{4t}\right)\right. \\ \left. - \left(1-\frac{(x-2)^2}{2t}\right)\exp\left(-\frac{(x-2)^2}{4t}\right)\right]. \end{aligned} \tag{5.3.24}$$

Since the function

$$f(\alpha) = (1 - 2\alpha)\exp(-\alpha)$$

is increasing for $\alpha > 3/2$, we see that there exists some $\delta_0 > 0$ such that

$$\frac{\partial}{\partial t} G_1(x, 0, t - \tau) > 0; \quad \frac{\partial}{\partial x} G_2(x, 1, t - \tau) > 0, \tag{5.3.25}$$

for $1 - \delta_0 < x < 1$ and $0 < \tau \leqq t < \delta_0$. Therefore, using the theorem of the mean, the positivity of $u(0, t)$, and the negativity of $q(1 - 0, t)$ for $t > 0$ sufficiently small, we find that

$$q(x, t) < G_1(x, 0, t) - m \int_0^t \frac{\partial}{\partial x} G_2(x, 1, t - \tau)\, d\tau. \tag{5.3.26}$$

Here $m = \min|q(1 - 0, t)|$ for $0 < \tau < \delta_0$. But

$$\int_0^t \frac{\partial}{\partial x} G_2(x, 1, t - \tau)\, d\tau = \operatorname{erfc}\frac{1 - x}{2\sqrt{t}};$$
$$G_1(x, 0, t) < \frac{\exp(-x^2/4t)}{2\sqrt{\pi t}}. \tag{5.3.27}$$

Since

$$\lim_{t \to 0} \sqrt{t} \exp\left(\frac{(1 - \delta)^2}{4t}\right) \operatorname{erfc}\frac{\delta}{2\sqrt{t}} = \infty \quad \text{for } \delta < \frac{1}{2},$$

we see that for $\delta_0 > 0$ sufficiently small

$$q(x, t) < 0 \text{ for } 1 - \delta_0 < x < 1,\ 0 < t < \delta_0. \tag{5.3.28}$$

But this shows that the inequality (5.1.14) is valid everywhere on Γ_η for $\eta > 0$ sufficiently small, as was to be proved.

§4. Existence of the solution

We will now prove the existence of a solution to problem (5.1.1)—(5.1.4). This will be obtained on the basis of the following lemmas.

LEMMA 12. *Let $\gamma > 0$ be a root of equation* (5.2.13) *and let γ_1 and γ_2 be arbitrary positive numbers such that $\gamma_1 < \gamma < \gamma_2$. Further, let z_j and $U_j(x, t)$ be defined by the conditions*

$$z_j(t) = 1 + 2\gamma_j a\sqrt{t} \qquad (j = 1, 2), \tag{5.4.1}$$

$$(5.4.2)\qquad \begin{cases} a^2(x)\dfrac{\partial^2 U_j}{\partial x^2} = \dfrac{\partial U_j}{\partial t},\ 0 < x < z_j(t),\ t > 0,\ x \neq 1; \\ \dfrac{\partial U_j}{\partial x}\Big|_{x=0} = 0; \quad U_j|_{x=z_j} = 0,\ t > 0; \quad U_j|_{t=0} = 1,\ 0 < x < 1; \\ U_j|_{x=1-0} = U_j|_{x=1+0}; \quad k_1 \dfrac{\partial U_j}{\partial x}\Big|_{x=1-0} = k_2 \dfrac{\partial U_j}{\partial x}\Big|_{x=1+0}. \end{cases}$$

Then there exists a $T > 0$ *such that*

$$(5.4.3)\qquad \dot{z}_1(t) < -\alpha + \beta \frac{\partial U_1}{\partial x}\Big|_{x=z_1(t)}; \quad \dot{z}_2(t) > -\alpha + \beta \frac{\partial U_2}{\partial x}\Big|_{x=z_2(t)}; \qquad 0 < t \leqq T.$$

LEMMA 13. *Let* $T > 0$ *and* $z_1(t)$ *be defined according to* (5.4.3) *and* (5.4.1), *respectively. Consider the monotonically decreasing sequence*

$$(5.4.4)\qquad t_0 > t_1 > \cdots > t_n > \cdots, \ \lim_{n\to\infty} t_n = 0; \quad t_0 < T.$$

Then for any $n \geqq 0$ *and for every* $t \in (0, T]$ *there exists a unique solution* $u_n(x,t)$, $y_n(t)$ *to the problem*

$$(5.4.5)\qquad \begin{cases} a^2(x)\dfrac{\partial^2 u_n}{\partial x^2} = \dfrac{\partial u_n}{\partial t},\ 0 < x < y_n(t),\ t > 0,\ x \neq 1; \\ \dfrac{\partial u_n}{\partial x}\Big|_{x=0} = 0; \quad u_n|_{x=y_n(t)} = 0; \quad t > 0; \quad u_n|_{t=0} = 1,\ 0 < x < 1; \\ u_n|_{x=1-0} = u_n|_{x=1+0}; \quad k_1 \dfrac{\partial u_n}{\partial x}\Big|_{x=1-0} = k_2 \dfrac{\partial u_n}{\partial x}\Big|_{x=1+0}; \quad t > 0, \end{cases}$$

$$(5.4.6)\qquad \dot{y}_n(t) = -\alpha + \beta \frac{\partial u_n}{\partial x}\Big|_{x=y_n(t)}, \quad t > t_n;$$

$$y_n(t) \equiv z_1(t), \quad 0 \leqq t \leqq t_n,$$

and moreover $\dot{y}_n(t)$ *is differentiable for* $t > t_n$.

LEMMA 14. *Let* u_n *and* y_n *be defined by the conditions of Lemma* 13 *and* z_1 *by relation* (5.4.1). *There exists a* $T_0 \in (0, T]$ *such that if* $0 < t \leqq T_0$ *then*

$$(5.4.7)\qquad \dot{y}_n(t) > 0; \quad -\frac{1}{z_1(t) - 1} \leqq -\frac{1}{y_n(t) - 1} < \frac{\partial u_n}{\partial x}\Big|_{x=y_n(t)} < 0$$

for arbitrary $n \geqq 0$.

Assume that Lemmas 12—14 are proved. We first note that, by virtue of the uniqueness of the solution of problem (5.4.5), (5.4.6) and hence of problem (5.4.2),

$$u_n \equiv U_1 \text{ for } 0 < x < z_1(t),\ 0 \leqq t \leqq t_n. \tag{5.4.8}$$

Thus from (5.4.3) and (5.4.6) it follows that

$$\dot{y}_n(t_n) > \dot{z}_1(t_n);\quad y_n(t_n) = z_1(t_n). \tag{5.4.9}$$

But this shows that on some interval $(t_n, t_n + \delta_n)$ we automatically have

$$z_1(t) < y_n(t) < z_2(t). \tag{5.4.10}$$

At the same time from the second equation (5.4.6) there exists some interval $(t_{n+1}, t_{n+1} + \delta_n^*)$ such that

$$y_{n+1}(t) > y_n(t) \text{ for } t_{n+1} < t < t_{n+1} + \delta_n^*. \tag{5.4.11}$$

We now assume that the set $\mathscr{S}_n$ of all $t \in (0, T]$ for which $y_n > y_{n+1}$ is not empty. Then, reasoning as in the proof of Lemma 1 in Chapter II, we find that if $\tau_n = \inf \mathscr{S}_n$, then the point $(y_n(\tau_n), \tau_n)$ is a zero, and yields a minimum boundary value of the function $u_{n+1} - u_n$. Since y_n is twice differentiable for $t > t_n$, the theorem of Friedman is applicable in the domain $0 < x < y_n(t)$. Consequently

$$\frac{\partial}{\partial x}(u_{n+1} - u_n)\Big|_{x=y_n(t),\, t=\tau_n} > 0. \tag{5.4.12}$$

Comparing (5.4.12) and (5.4.6) and noting that $\beta < 0$, we find that

$$\dot{y}_{n+1} - \dot{y}_n > 0 \text{ for } t = \tau_n, \tag{5.4.13}$$

which contradicts the definition of τ_n. Thus

$$y_{n+1}(t) \geqq y_n(t) \text{ for } 0 < t \leqq T. \tag{5.4.14}$$

In exactly the same way, we prove that for every n

$$z_1(t) \leqq y_n(t) \leqq z_2(t) \text{ for } 0 < t \leqq T. \tag{5.4.15}$$

We note that this conclusion is not based on Lemma 14.

Thus the sequence $\{y_n(t)\}$ is monotonic and bounded. In addition it is equicontinuous, since by virtue of (5.4.7) and (5.4.6) the sequence $\{\sqrt{t}\,\dot{y}_n(t)\}$ is uniformly bounded. Consequently $\{y_n(t)\}$ is uniformly convergent to some function $y(t)$ satisfying the Lipschitz condition

$$|y(t) - y(\tau)| < \frac{A|t-\tau|}{\sqrt{t}+\sqrt{\tau}}, \quad 0 < \tau, t \leqq T_0. \tag{5.4.16}$$

From the monotone convergence of $\{y_n\}$ to y and from the maximum principle, it follows that $\{u_n(x,t)\}$ is uniformly convergent to a bounded solution $U(x,t)$ of the problem

$$\begin{gathered} a^2(x)\frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t}, \ 0 < x < y(t), \ x \neq 1, \ 0 < t \leqq T_0; \\ \left.\frac{\partial U}{\partial x}\right|_{x=0} = 0; \quad U|_{x=y(t)} = 0; \quad U|_{t=0} = 1; \\ U|_{x=1-0} = U|_{x=1+0}; \quad k_1 \left.\frac{\partial U}{\partial x}\right|_{x=1-0} = k_2 \left.\frac{\partial U}{\partial x}\right|_{x=1+0}. \end{gathered} \tag{5.4.17}$$

Here, since $y(t)$ satisfies the Lipschitz condition (5.4.16), it follows that the limit

$$\lim_{x \to y(t)} \frac{\partial U}{\partial x} = V(t), \ t > 0, \tag{5.4.18}$$

exists, and moreover that $V(t)$ is continuous for $t > 0$.

We now show that

$$y(t) = 1 - \alpha t + \beta \int_0^t V(\tau)\, d\tau, \tag{5.4.19}$$

i.e. that $U(x,t)$, $y(t)$ satisfies every condition of the problem (5.1.1)—(5.1.4). In fact, the first of the equations (5.4.17) implies the integral identity

$$\int_0^t d\tau \int_0^{y(\tau)} k(x) \frac{\partial^2 U}{\partial x^2}\, dx = \int_0^t d\tau \int_0^{y(\tau)} \frac{k(x)}{a^2(x)} \frac{\partial U}{\partial \tau}\, dx. \tag{5.4.20}$$

Carrying out the integration, with the notation (5.4.18) and condition (5.4.17), we find that

$$k_2 \int_0^t V(\tau)\, d\tau = k_1 \int_0^1 (U(x,t) - 1)\, dx + \frac{k_2}{a^2} \int_1^{y(t)} U(x,t)\, dx. \tag{5.4.21}$$

On the other hand, from (5.4.5) it follows analogously that

$$\begin{aligned} k_2 \int_0^t \left.\frac{\partial u_n}{\partial x}\right|_{x=y_n(\tau)} d\tau &= k_1 \int_0^1 (u_n(x,t) - 1)\, dx \\ &\quad + \frac{k_2}{a^2} \int_1^{y_n(t)} u_n(x,t)\, dx. \end{aligned} \tag{5.4.22}$$

Using (5.4.6), it is possible to rewrite (5.4.22) in the form

$$k_2 \int_0^{t_n} \frac{\partial U_1}{\partial x}\bigg|_{x=y_n(\tau)} d\tau + \frac{k_2}{\beta}(y_n(t) + \alpha t) - \frac{k_2}{\beta}[z_1(t_n) + \alpha t_n]$$

(5.4.23)
$$= k_1 \int_0^1 (u_n(x,t) - 1)\,dx + \frac{k_2}{a^2}\int_1^{y_n(t)} u_n(x,t)\,dx.$$

Passing to the limit for $n \to \infty$ and using the uniform convergence of y_n to y and u_n to U, we obtain

$$\frac{k_2}{\beta}(y(t) + \alpha t - 1) = k_1 \int_0^1 (U(x,t) - 1)\,dx + \frac{k_2}{a^2}\int_1^{y(t)} U(x,t)\,dx. \tag{5.4.24}$$

Comparing (5.4.24) and (5.4.21) leads to (5.4.19), which completes the proof.

We note that from (5.4.19) we obtain the differentiability of y, since V is continuous for $t > 0$.

Thus for the proof of existence of the solution it is sufficient to prove Lemmas 12—14.

The validity of Lemma 12 follows at once from the analysis in §2. In fact, repeating the derivation of equation (5.2.11) and using the notation

$$W_i(t) = \sqrt{t}\frac{\partial}{\partial x} U_i\bigg|_{x=z_i(t)} \qquad (i = 1, 2), \tag{5.4.25}$$

we find that

$$W_i(0) = -\frac{2\exp(-\gamma_i^2)}{(1+\lambda)a\sqrt{\pi}} - W_i(0)(\operatorname{erf}\gamma_i - \delta \operatorname{erfc}\gamma_i),$$

or in the same way, by virtue of (5.2.9)

$$W_i(0) = -\frac{\exp(-\gamma_i^2)}{(\lambda + \operatorname{erf}\gamma_i)a\sqrt{\pi}} \equiv f(\gamma_i). \tag{5.4.26}$$

But $\dot{f}(\gamma) > 0$ for $\gamma > 0$. Using this and the inequality $\gamma_1 < \gamma < \gamma_2$, we obtain

$$W_1(0) < w(0) < W_2(0), \tag{5.4.27}$$

which by virtue of (5.2.11*) and $\beta < 0$ gives

$$\beta W_2(0) < \gamma a < \beta W_1(0). \tag{5.4.28}$$

This means, however, that

$$\lim_{t\to 0}\sqrt{t}\beta\frac{\partial U_2}{\partial x}\bigg|_{x=z_2(t)} < \lim_{t\to 0}\sqrt{t}\dot{z}_2(t);$$
$$\lim_{t\to 0}\sqrt{t}\beta\frac{\partial U_1}{\partial x}\bigg|_{x=z_1(t)} > \lim_{t\to 0}\sqrt{t}\dot{z}_1(t), \tag{5.4.29}$$

and that there exists an interval $(0, T)$ on which the inequality (5.4.3) is automatically valid.

We shall prove the validity of Lemma 14, assuming that of Lemma 13. With this aim we note the following. If u_n is a uniformly bounded solution of problem (5.4.5), then u_n obeys the strong maximum principle, from which, by virtue of the theorem of Friedman, it follows that

$$0 \leqq u_n(1,t) < 1 \text{ for } 0 \leqq t \leqq T. \tag{5.4.30}$$

We assume that $\dot{y}_n > 0$ on the interval $0 < t < T_n$, and we examine the function

$$\overline{U}_n(x,t) = 1 - \frac{x-1}{y_n(t)-1}, \quad 1 < x < y_n(t),\ 0 < t \leqq T_n. \tag{5.4.31}$$

In the indicated domain $\overline{U}_n$ is superparabolic, reducing to zero for $x = y_n(t)$ and to 1 for $x = 1$. Consequently, by the maximum principle

$$\overline{U}_n \geqq u_n \text{ for } 1 < x < y_n(t), \quad 0 < t \leqq T_n, \tag{5.4.32}$$

with the equality sign automatically valid for $x = y_n(t)$. But this means that

$$\frac{\partial \overline{U}_n}{\partial x} < \frac{\partial u_n}{\partial x}\bigg|_{x=y_n(t)} < 0, \tag{5.4.33*}$$

i.e.

$$-\frac{1}{y_n(t)-1} < \frac{\partial u_n}{\partial x}\bigg|_{x=y_n(t)} < 0. \tag{5.4.33}$$

Now, using the inequality (5.4.15) proved above without referring to Lemma 14, we see that the second of the assertions (5.4.7) holds, provided

$$\tau^* = \operatorname*{Inf}_n T_n > 0. \tag{5.4.34}$$

For the proof of (5.4.34) we note that, by (5.2.3), for $0 \leqq t \leqq t_n$

$$u_n(x,t) \equiv U_1(x,t) = \int_0^t G(x,\xi,t)\,d\xi + a^2\int_0^t V_1(\tau)\,G(x,z_1(\tau),t-\tau)\,d\tau. \tag{5.4.35}$$

From the definition (S2.2.17) of Supplement 2 it follows that

$$G(1,\xi,t) = \begin{cases} \dfrac{1+\delta}{\sqrt{\pi t}} \exp\left(-\dfrac{(1-\xi)^2}{4t}\right) + o\left(\exp\left(-\dfrac{\kappa}{t}\right)\right), & \xi < 1 \\ \dfrac{1-\delta}{a\sqrt{\pi t}} \exp\left(-\dfrac{(1-\xi)^2}{4a^2 t}\right) + o\left(\exp\left(-\dfrac{\kappa}{t}\right)\right), & \xi > 1, \end{cases} \tag{5.4.36}$$

where $\kappa > 0$. Inserting (5.4.36) into (5.4.35), we see that

$$V_1(t) = W_1(t)/\sqrt{t}, \tag{5.4.37}$$

where $W_1(0)$ is defined in agreement with (5.4.26), and, passing to the limit for $t \to 0$, we obtain

$$\lim_{t\to 0} U_1(1,t) = (1+\delta) + a\frac{W_1(0)}{\sqrt{\omega}} \int_0^1 \frac{1-\delta}{\sqrt{x}} \frac{\exp\left(-\gamma_1^2 \dfrac{x}{1-x}\right)}{\sqrt{1-x}}\, dx. \tag{5.4.38}$$

Computing the integral on the right, we find that

$$\lim_{t\to 0} U_1(1,t) = 1 + \delta + (1-\delta) a\sqrt{\pi}\, W_1(0) \exp(\gamma_1^2) \operatorname{erfc} \gamma_1.$$

Inserting $W_1(0)$ from (5.4.26), we obtain

$$\lim_{t\to 0} U_1(1,t) = 1 + \delta + (1-\delta) \frac{\operatorname{erfc} \gamma_1}{\lambda + \operatorname{erf} \gamma_1}. \tag{5.4.39}$$

But by definition (5.2.9) we have

$$1 - \delta = \frac{2\lambda}{1+\lambda}; \quad 1 + \delta = \frac{2}{1+\lambda}; \quad \lambda > 0. \tag{5.4.40}$$

Thus

$$\lim_{t\to 0} U_1(1,t) = (1+\delta)\left[1 - \frac{1 - \operatorname{erf} \gamma_1}{1 + (1/\lambda) \operatorname{erf} \gamma_1}\right] > 0. \tag{5.4.41}$$

Consequently by continuity there exists some $T_1 > 0$ such that

$$U_1(1,t) \geqq m > 0 \text{ for } 0 \leqq t \leqq T_1. \tag{5.4.42}$$

But by the proof above

$$u_n \geqq U_1 \tag{5.4.43}$$

for arbitrary $n \geqq 0$. Consequently, together with (5.4.42) we have the

inequality

$$u_n(1,t) \geqq m > 0 \text{ for } 0 \leqq t \leqq T_1 \leqq T, \tag{5.4.44}$$

with $T_1 > 0$ independent of the choice of n.

We now choose

$$k > -\beta/a^2 \tag{5.4.45}$$

and consider the function

$$\underline{U}_n(x,t) = m \exp(-k)\left[\exp\left(k - k\frac{x-1}{y_n - 1}\right) - 1\right]. \tag{5.4.46}$$

It is easily seen that $\underline{U}_n$ is superparabolic for $1 < x < y_n(t)$ and $0 < t \leqq T_n$. In fact,

$$\operatorname{sgn} L_2 \underline{U}_n = \operatorname{sgn}[a^2 k - (x-1)\dot{y}_n(t)]. \tag{5.4.47}$$

But from (5.4.7), (5.4.6) and (5.4.3) it follows that for $0 \leqq t \leqq T_n$

$$(y_n(t) - 1)\dot{y}_n(t) \leqq \beta(y_n(t) - 1)\frac{\partial}{\partial x} u_n\Big|_{x=y_n(t)} \leqq -\beta. \tag{5.4.48}$$

Comparing (5.4.48), (5.4.47) and (5.4.45) proves our assertion.

Furthermore,

$$\begin{aligned} &\underline{U}_n|_{x=1} = m \exp(-k)[\exp k - 1] < m; \\ &\underline{U}_n|_{x=y_n(t)} = 0. \end{aligned} \tag{5.4.49}$$

Consequently for $0 \leqq t \leqq T_1$ the limiting value of $\underline{U}_n$ is less than that of u_n, whence from the uniform boundedness of $\underline{U}_n$ and u_n, the parabolicity of u_n, the superparabolicity of $\underline{U}_n$ and the maximum principle, it follows that

$$\underline{U}_n(x,t) \leqq u_n(x,t) \text{ for } 1 < x < y_n(t), \quad 0 < t \leqq \min(T_1, T_n). \tag{5.4.50}$$

Moreover

$$\underline{U}_n = u_n = 0 \text{ for } x = y_n(t), \quad t > 0, \tag{5.4.51}$$

whence it follows that

$$\frac{\partial}{\partial x} u_n\Big|_{x=y_n(t)} \leqq \frac{\partial}{\partial x} \underline{U}_n\Big|_{x=y_n(t)} \text{ for } 0 < t \leqq \min(T_1, T_n), \tag{5.4.52}$$

or equivalently

$$\left.\frac{\partial}{\partial x} u_n\right|_{x=y_n(t)} < -\frac{mk}{y_n(t)-1} \text{ for } 0 < t \leqq \min(T_1, T_n). \tag{5.4.53}$$

Majorizing this inequality by the use of (5.4.15), we find that

$$\left.\frac{\partial}{\partial x} u_n\right|_{x=y_n(t)} < -\frac{m_k}{z_2(t)-1} \text{ for } 0 < t \leqq \min(T_1, T_n). \tag{5.4.54}$$

Finally, introducing (5.4.54) into the first of the conditions (5.4.6), we obtain

$$\dot{y}_n(t) > -\alpha - \frac{mk\beta}{z_2(t)-1} \equiv f(t) \text{ for } 0 < t \leqq \min(T_1, T_n). \tag{5.4.55}$$

Now let T_2 be defined by the condition $f(t) > 0$ for $0 \leqq t \leqq T_2$. Since $z_2(0) = 0$, we see that $T_2 > 0$. But T_2 and T_1 are defined independently of n, whence it is obvious that $T_n \geqq T_2$. Consequently T_n can be defined by the equation

$$T_n = \min(T_1, T_2), \tag{5.4.56}$$

which shows that

$$\tau^* = \min(T_1, T_2) > 0. \tag{5.4.57}$$

Lemma 14 is thus proved.

It now remains for us only to prove Lemma 13. But its proof is obviously obtained by a repetition of the analysis carried out above for problems B_1 and A_1.

In fact, we determine $W_1(t)$ as the solution of equation (5.2.6), if in it we write y in place of z_1. The possibility of constructing the solution of this equation for $t > 0$ sufficiently small by use of the Picard iteration process is easily established with the help of estimates similar to those obtained in §3 of Chapter III for the investigation of equation $(3.3.4_2)$.[3]

Using the solution of equation (5.2.6), we calculate $U_1(x,t)$ with the help of the quadrature (5.2.3). After this the determination of u_n and y_n is based on that of the completely similar problem B_1. In fact, u_n is defined by the quadrature

[3] We note that the Green's function G appearing in (5.2.6) differs from the Green's function appearing in $(3.3.4_2)$ by terms tending to zero for $t \to 0$ faster than any power of t; and, moreover, in the limit the term $E(x+\xi-2, a^2t)$ has a coefficient δ which is less than unity in absolute value. Therefore the convergence proof for the iteration process which was used in §3 of Chapter III remains valid in the case of equation (5.2.6).

$$u_n(x,t) = \int_0^{y_n(t_n)} G(x,\xi,t-t_n)\,U_1(\xi,t_n)\,d\xi + a^2\int_{t_n}^{t} v_n(\tau)\,G(x,y_n(\tau),t-\tau)\,d\tau, \tag{5.4.58}$$

where

$$v_n(t) = \frac{\partial}{\partial x}\,u_n\Big|_{x=y_n(t)}. \tag{5.4.59}$$

Hence it follows at once that, as in §2,

$$v_n(t) = 2\int_0^{y_n(t_n)} \frac{\partial}{\partial \xi}\,U_1(\xi,t_n)g(y_n(t),\xi,t-t_n)\,d\xi + 2a^2\int_{t_n}^{t} v_n(\tau)\,\frac{\partial}{\partial x}\,G(y_n(t),y_n(\tau),t-\tau)\,d\tau. \tag{5.4.60}$$

Rewriting (5.4.6) in the form

$$y_n(t) = z_1(t_n) - \alpha(t-t_n) + \beta\int_{t_n}^{t} v_n(\tau)\,d\tau, \tag{5.4.61}$$

we obtain a system of integral equations which does not essentially differ from the system of integral equations of problem B_1. Using a method of continuation of a solution similar to that used in Chapter II for the construction of a solution to problem B_1 "in the large", we prove the existence of a solution on any interval $(0,T)$. For this, using the a priori estimate

$$0 < m = u_n(1,t) \leqq 1 \quad (0 \leqq t \leqq T) \tag{5.4.62}$$

and the inequality (5.4.15), we can completely carry over the investigation of §2, Chapter II to the problem (5.4.5), (5.4.6), since the continuation of the solution to any interval $(0,T)$ does not require a special proof. Thus Lemma 13, and with it Theorem 17, is valid.

REMARK. Since $\gamma_1 < \gamma$ in the definition (5.4.1) can be arbitrarily close to γ, it is possible to take

$$z_1(t) = 1 - \alpha t + 2\gamma\sqrt{t}. \tag{5.4.63}$$

in definition (5.4.2). In this connection, for practical computation we can recommend the algorithm described in §6 of Chapter I, with the single change that on the interval $(0,t_0)$ the iteration is not carried out but instead we take

(5.4.64) $$w(t) \equiv w(0); \quad y(t) \equiv 1 - \alpha t + 2\gamma\sqrt{t}.$$

For more details we refer the reader to Supplement 3, §2.

§5. Investigation of the initial melting process

Up to now we have examined the process of crystallization of a melt upon insertion of a plate, under the assumption that its fusion temperature is below the temperature of crystallization of the melt. We now examine the case of insertion in the melt of a plate of less refractory material. As we noted in Chapter I, §3, Part 1, in this case it is a priori possible that the course of development of the process will be like one of the following:

A. Crystallization of the melt and melting of the plate are initiated simultaneously at the moment of insertion.

B. Melting of the plate precedes crystallization of the melt.

C. Crystallization of the melt precedes melting of the plate.

Finally, we assume that the process runs its course without superheating or supercooling.

Our problem is now to investigate what really occurs in the indicated processes.

As in the first part, we will assume that the jump of density for the phase change can be ignored, as well as the thermal expansion of each phase.

For later use we establish the relations between the dimensionless parameters used in the investigation and the dimensioned parameters entering in the original problem. We will use the indices $i = 1, 2, 3$ for the material of the plate in solid and liquid phases and the material of the melt in the solid phase respectively. We mark the dimensioned coordinate, the coefficients of heat conduction, temperature conductivity, latent heat and density by the sign $\sim$ and introduce the dimensionless variables by setting

$$x = \frac{\tilde{x}}{l}; \quad t = \frac{\tilde{a}_1^2 \tilde{t}}{l^2}; \quad u = \frac{T - T_k}{T_\theta - T_k}; \quad \vartheta = \frac{T_n - T_k}{T_0 - T_k};$$

(5.5.1)
$$a_i^2 = \frac{\tilde{a}_i^2}{\tilde{a}_1^2}; \quad k_i = \frac{\tilde{k}_i}{\tilde{k}_1}; \quad \lambda_i = \frac{k_3 a_i}{k_i a_3}; \quad \delta_i = \frac{1 - \lambda_i}{1 + \lambda_i};$$

$$\beta_i = \frac{\tilde{k}_i (T_0 - T_k)}{\tilde{\gamma}_i \tilde{\rho}_i \tilde{a}_1^2}; \quad \mu_i = \frac{\tilde{k}_i (T_0 - T_k)}{\tilde{\gamma}_1 \tilde{\rho}_1 \tilde{a}_1^2}; \quad \alpha_i = \frac{ql}{\tilde{\gamma}_i \tilde{\rho}_i \tilde{a}_1^2}.$$

We note that γ_i denotes the specific latent heat of the phase change, ρ_i the density of the phase, $2l$ the thickness of the plate, T_k the temperature of crystallization of the melt, T_n the temperature of melting of the plate, and T_0 its initial temperature, where in addition

(5.5.1*) $$T_0 < T_n < T_k.$$

We assume that $u = 0$ is the dimensionless temperature of crystallization of the melt, and that $u = \vartheta$, $0 < \vartheta < 1$, is the dimensionless melting temperature of the plate.

In the notation (5.5.1) the initial state of each of the possible cases described above is given by the following conditions.

A. *Crystallization of the melt and melting of the plate, initiated simultaneously.*

(5.5.2$_1$) $$a^2(x)\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},\ 0 < x < y(t),\ x \neq z(t),\ x \neq 1;\quad t > 0;$$

(5.5.2$_2$) $$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0;\quad u|_{x=z(t)\pm 0} = \vartheta;$$

$$u|_{x=y(t)-0} = 0;\quad t > 0;\quad u|_{t=0} = 1;\quad 0 < x < 1;$$

(5.5.2$_3$) $$u|_{x=1-0} = u|_{x=1+0};\quad k_2\left.\frac{\partial u}{\partial x}\right|_{x=1-0} = k_3\left.\frac{\partial u}{\partial x}\right|_{x=1+0},\quad t > 0;$$

(5.5.2$_4$) $$\dot{z}(t) = \mu_1\left.\frac{\partial u}{\partial x}\right|_{x=z(t)-0} - \mu_2\left.\frac{\partial u}{\partial x}\right|_{x=z(t)+0},\ t > 0,\quad z(0) = 1;$$

(5.5.2$_5$) $$\dot{y}(t) = -\alpha_3 + \beta_3\left.\frac{\partial u}{\partial x}\right|_{x=y(t)-0},\quad t > 0,\quad y(0) = 1.$$

Here

(5.5.2$_6$) $$a^2(x) = \begin{cases} a_1^2 = 1, & 0 < x < z(t), \\ a_2^2, & z(t) < x < 1, \\ a_3^2, & 1 < x < y(t), \end{cases}$$

$x = z(t)$ is the equation of the melting front of the plate, and $x = y(t)$ is the equation of the crystallization front of the melt.

B. *Melting of the plate preceding crystallization of the melt.*

(5.5.3$_1$) $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},\quad 0 < x < z(t),\ t > 0,$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0; \quad u|_{x=z(t)} = \vartheta, \; t>0; \quad u|_{t=0} = 1, \; 0<x<1; \tag{5.5.3$_2$}$$

$$\dot{z}(t) = -\alpha_1 + \beta_1 \left.\frac{\partial u}{\partial x}\right|_{x=z(t)}, \; t>0 \; z(0)=1. \tag{5.5.3$_3$}$$

In agreement with physical logic of the problem the process B is possible if $z(t)<1$ in a neighborhood of $t=0$.

C. *Crystallization of the melt preceding melting of the plate.*

$$a^2(x)\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \; 0<x<y(t), \; t>0, \; x \neq 1, \tag{5.5.4$_1$}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0; \quad u|_{x=y(t)} = 0; \quad t>0; \quad u|_{t=0} = 1, \; 0<x<1, \tag{5.5.4$_2$}$$

$$u|_{x=1-0} = u|_{x=1+0}; \quad k_1 \left.\frac{\partial u}{\partial x}\right|_{x=1-0} = k_3 \left.\frac{\partial u}{\partial x}\right|_{x=1+0}, \quad t>0, \tag{5.5.4$_3$}$$

$$\dot{y}(t) = -\alpha_3 + \beta_3 \left.\frac{\partial u}{\partial x}\right|_{x=y(t)}; \quad y(0)=1. \tag{5.5.4$_4$}$$

Here

$$a^2(x) = \begin{cases} a_1^2 = 1, & 0<x<1 \\ a_3^2, & 1<x<y(t). \end{cases} \tag{5.5.4$_5$}$$

The process C can occur only in the case when in addition to condition (5.5.4$_i$) $(i=1,2,3,4)$ we have the condition

$$u(1,t) > \vartheta, \quad t>0. \tag{5.5.4$_6$}$$

We will first examine the process A, using the notation

$$u|_{x<z(t)} = u_1; \quad \left.\frac{\partial u}{\partial x}\right|_{x=z-0} = v_1(t); \quad \left.\frac{\partial u}{\partial x}\right|_{x=z+0} = v_2(t); \quad \left.\frac{\partial u}{\partial x}\right|_{x=y-0} = v_3(t); \tag{5.5.5}$$

$$\sqrt{t}\, v_i(t) = w_i(t) \quad (i=1,2,3);$$

$$\lim_{t\to 0}\sqrt{t}\,\dot{z}(t) = -\gamma_1; \quad \lim_{t\to 0}\sqrt{t}\,\dot{y}(t) = \gamma_3.$$

We derive the integral equations of problem (5.5.2$_i$) $(i=1,\cdots,5)$.

Repeating the derivations of Chapters I and II, we find that

$$u_1 = \vartheta + (1-\vartheta)\int_0^1 g_2(x,\xi,t)\,d\xi + \int_0^t v_1(\tau)g_2(x,z(\tau),t-\tau)\,d\tau, \tag{5.5.6}$$

whence

$$\begin{aligned} w_1(t) = &-2(1-\vartheta)\sqrt{t}\,g_1(z(t),1,t) \\ &-2\int_0^t w_1(\tau)\sqrt{\frac{t}{\tau}}\frac{\partial}{\partial\xi}g_1(z(t),z(\tau),t-\tau)\,d\tau. \end{aligned} \tag{5.5.7}$$

Here g_1 and g_2 are Green's functions of the first and second boundary problems of the equation $L_1 u = 0$, defined in agreement with (0.2.5) and (0.2.6).

Passing to the limit for $t \to 0$ in (5.5.7) and using (5.5.5), we obtain

$$w_1(0) = -\frac{1-\vartheta}{\sqrt{\pi}}\frac{\exp(-\gamma_1^2)}{\operatorname{erfc}\gamma_1}. \tag{5.5.8}$$

Let us evaluate $w_i(0)$ $(i = 2, 3)$. For this we use the fundamental solution g of the heat equation for a double layer, and the adjoint function g^* constructed in §2 of Supplement 2:

$$g(x,\xi,t) = \begin{cases} E(x-\xi,a_2^2t) + \delta_2 E(x+\xi-2,a_2^2t) & \text{for } x,\xi<1 \\ \dfrac{1-\delta_2}{a_3}E\left(\dfrac{x-1}{a_2}-\dfrac{\xi-1}{a_3},t\right) & \text{for } x<1<\xi \\ \dfrac{1+\delta_2}{a_2}E\left(\dfrac{x-1}{a_3}-\dfrac{\xi-1}{a_2},t\right) & \text{for } \xi<1<x \\ E(x-\xi,a_3^2t) - \delta_2 E(x+\xi-2,a_3^2t) & \text{for } 1<x,\xi, \end{cases} \tag{5.5.9}$$

$$g^*(x,\xi,t) = \begin{cases} E(x-\xi,a_2^2t) - \delta_2 E(x+\xi-2,a_2^2t) & \text{for } x,\xi<1 \\ \dfrac{1-\delta_2}{a_2}E\left(\dfrac{x-1}{a_2}-\dfrac{\xi-1}{a_3},t\right) & \text{for } x<1<\xi \\ \dfrac{1+\delta_2}{a_3}E\left(\dfrac{x-1}{a_3}-\dfrac{\xi-1}{a_2},t\right) & \text{for } \xi<1<x \\ E(x-\xi,a_3^2t) + \delta_2 E(x+\xi-2,a_3^2t) & \text{for } 1<x,\xi. \end{cases} \tag{5.5.10}$$

As in §2, we find for $z(t) < x < y(t)$ that

$$
\begin{aligned}
u(x,t) = &- a_2^2 \int_0^t v_2(\tau) g(x, z(\tau), t-\tau)\, d\tau \\
&+ \vartheta \int_0^t \left[a_2^2 \frac{\partial}{\partial \xi} - \dot{z}(\tau) \right] g(x, z(\tau), t-\tau)\, d\tau \\
&+ a_3^2 \int_0^t v_3(\tau) g(x, y(\tau), t-\tau)\, d\tau,
\end{aligned} \tag{5.5.11}
$$

whence

$$
\begin{aligned}
w_2(t) = &- 2\vartheta \sqrt{t}\, g^*(z(t), z(0), t) \\
&+ 2a_2^2 \int_0^t w_2(\tau) \sqrt{\frac{t}{\tau}} \frac{\partial}{\partial \xi} g^*(z(t), z(\tau), t-\tau)\, d\tau \\
&- 2a_3^2 \int_0^t w_3(\tau) \sqrt{\frac{t}{\tau}} \frac{\partial}{\partial \xi} g^*(z(t), y(\tau), t-\tau)\, d\tau,
\end{aligned} \tag{5.5.12}
$$

$$
\begin{aligned}
w_3(t) = &- 2\vartheta \sqrt{t}\, g^*(y(t), z(0), t) \\
&+ 2a_2^2 \int_0^t w_2(\tau) \sqrt{\frac{t}{\tau}} \frac{\partial}{\partial \xi} g^*(y(t), z(\tau), t-\tau)\, d\tau \\
&- 2a_3^2 \int_0^t w_3(\tau) \sqrt{\frac{t}{\tau}} \frac{\partial}{\partial \xi} g^*(y(t), y(\tau), t-\tau)\, d\tau.
\end{aligned} \tag{5.5.13}
$$

Substituting g^* from (5.5.10), passing to the limit for $t \to 0$, using (5.5.5) and the initial condition $z(0) = 1 - 0$, $u(0) = 1$, and taking (5.2.12) into account, we obtain

$$
\begin{aligned}
w_2(0) = &- \frac{\vartheta(1-\delta_2)}{a_2\sqrt{\pi}} \exp(-\xi^2) - w_2(0)\,[\operatorname{erf}\xi + \delta_2 \operatorname{erfc}\xi] \\
&+ w_3(0)(1-\delta_2)\frac{a_3}{a_2} I(\xi, \eta),
\end{aligned} \tag{5.5.14}
$$

$$
\begin{aligned}
w_3(0) = &- \frac{\vartheta(1+\delta_2)}{a_3\sqrt{\pi}} \exp(-\eta^2) + w_2(0)(1+\delta_2)\frac{a_2}{a_3} I(\eta, \xi) \\
&- w_3(0)\,[\operatorname{erf}\eta - \delta_2 \operatorname{erfc}\eta\,],
\end{aligned} \tag{5.5.15}
$$

where

$$
\xi = \gamma_1/a_2; \qquad \eta = \gamma_3/a_3 \tag{5.5.16}
$$

and

$$(5.5.17)\qquad I(\xi,\eta)=\frac{1}{\sqrt{\pi}}\int_0^1(\xi+\eta\sqrt{x})\,\frac{\exp\left\{-(\xi+\eta\sqrt{x})^2\dfrac{1}{1-x}\right\}}{\sqrt{x}(1-x)^{3/2}}\,dx.$$

The substitution

$$\frac{\xi+\eta\sqrt{x}}{\sqrt{1-x}}=z$$

reduces the expression $I(\xi,\eta)$ to the form

$$(5.5.17^*)\qquad I(\xi,\eta)=\frac{2}{\sqrt{\pi}}\int_\xi^\infty\frac{z\exp(-z^2)\,dz}{\sqrt{\eta^2-\xi^2+z^2}}.$$

Thus

$$(5.5.17^{**})\qquad I(\xi,\eta)=\exp(\eta^2-\xi^2)\operatorname{erfc}\eta.$$

We now observe that from (5.5.16), (5.5.5) and the conditions $(5.5.2_4)$ and $(5.5.2_5)$ it follows that

$$(5.5.18)\qquad \xi=\frac{\mu_2}{a_2}w_2(0)-\frac{\mu_1}{a_2}w_1(0);\qquad \eta=\frac{\beta_3}{a_3}w_3(0).$$

But at the same time, by virtue of (5.5.16), equation (5.5.8) can be written in the form

$$(5.5.19)\qquad w_1(0)=-\frac{1-\vartheta}{\sqrt{\pi}}\cdot\frac{\exp(-a_2^2\xi^2)}{\operatorname{erfc}(a_2\xi)}.$$

Eliminating $w_i(0)$ from the system (5.5.14), (5.5.15), (5.5.18) and (5.5.19) and using (5.5.17**), we obtain

$$(5.5.20)\qquad \eta\exp(\eta^2)=\frac{\beta_3a_2^2}{a_3^2\mu_2\lambda_2}\left[\xi-\frac{\mu_1}{a_2}\cdot\frac{1-\vartheta}{\sqrt{\pi}}\,\frac{\exp(\xi^2-a_2^2\xi^2)}{\operatorname{erfc}a_2\xi}\right];$$

$$(5.5.21)\qquad \xi=\frac{\mu_1}{a_2\sqrt{\pi}}\cdot\frac{(1-\vartheta)\exp(-a_2^2\xi^2)}{\operatorname{erfc}a_2\xi}-\frac{\mu_2\vartheta\exp(-\xi^2)}{a_2^2\sqrt{\pi}(\operatorname{erf}\eta+\lambda_2\operatorname{erf}\xi)}\equiv F(\xi).$$

Let η_0 be the root of the equation

$$\eta_0 \exp(\eta_0^2) = -\frac{\beta_3 \mu_1}{\mu_2 \lambda_2} \frac{a_2}{a_3^2} \frac{1-\vartheta}{\sqrt{\pi}}. \tag{5.5.22}$$

Equation (5.5.20) defines η as a monotonic function of ξ, varying from η_0 to ∞ for ξ varying from 0 to ∞. Hence (5.5.21) has one and only one positive root ξ if and only if

$$\sqrt{\pi} F(0) = \frac{\mu_1(1-\vartheta)}{a_2} - \frac{\mu_2 \vartheta}{\lambda_2 a_2^2 \operatorname{erf} \eta_0} > 0, \tag{5.5.23}$$

or equivalently, by virtue of (5.5.1), when

$$\lambda_1 > \frac{1-\vartheta}{\vartheta} \operatorname{erf} \eta_0. \tag{5.5.23*}$$

Thus the process A is possible only when we have (5.5.23).

We observe that the formulation of the conditions making possible the development of the process A was obtained only as a necessary condition. However, using the methods of the previous sections, it is possible to prove that under the necessary conditions found there the solution of problem $(5.5.2_1)$—$(5.5.2_5)$ exists and is unique. Thus the condition we have found for the process A is not only necessary but also sufficient.

We now examine the process B. In this case the equation (5.5.7) remains valid. Thus

$$\lim_{t \to 0} \frac{\partial u}{\partial x}\bigg|_{x=z(t)} = -\infty. \tag{5.5.24}$$

Since $\beta_1 < 0$ it follows from (5.5.3) that

$$\lim_{t \to 0} \dot{z}(t) = +\infty. \tag{5.5.25}$$

However, since $z(0) = 1$, this contradicts the condition $z(t) < 1$ for $t > 0$. Thus the development of the process as described by B is impossible.

We finally examine C. Noting that equations $(5.5.4_1)$—$(5.5.4_4)$ coincide with equations (5.1.5)—(5.1.4) up to notation for the parameters, we find that the solution to $(5.5.4_1)$—$(5.5.4_4)$ exists and is unique. Using (5.4.41) and (5.5.1), we find that

$$\lim_{t \to 0} u(1,t) = \frac{\operatorname{erf} \eta}{\lambda_1 + \operatorname{erf} \eta}. \tag{5.5.26}$$

Here η is a root of equation (5.2.13) which can be reduced to the form

$$\eta \exp(\eta^2) = -\frac{\beta_3}{\lambda_1 a_3^2 \sqrt{\pi}} \cdot \frac{\lambda_1}{\lambda_1 + \operatorname{erf}\eta}. \tag{5.5.27}$$

But this shows that the development of the process C is possible if

$$\frac{\operatorname{erf}\eta}{\lambda_1 + \operatorname{erf}\eta} > \vartheta, \tag{5.5.28}$$

and impossible if

$$\frac{\operatorname{erf}\eta}{\lambda_1 + \operatorname{erf}\eta} \leqq \vartheta. \tag{5.5.29}$$

It is easily seen that the conditions under which processes A and C are possible cannot coincide. In fact, let us assume the contrary. We note that (5.5.28) can be written in the form

$$\frac{\lambda_1}{\lambda_1 + \operatorname{erf}\eta} < 1 - \vartheta. \tag{5.5.30}$$

Comparing (5.5.30) and (5.5.23*), we find that

$$\eta > \eta_0. \tag{5.5.31}$$

On the other hand, from (5.5.27), (5.5.22) and (5.5.1) it follows that

$$\frac{\eta \exp(\eta^2)}{\eta_0 \exp(\eta_0^2)} = \frac{\lambda_1}{\lambda_1 + \operatorname{erf}\eta} \frac{1}{1 - \vartheta}. \tag{5.5.32}$$

Comparing (5.5.32) and (5.5.30), we obtain

$$\eta < \eta_0, \tag{5.5.33}$$

which contradicts (5.5.31). The contradiction proves the incompatibility of processes A and C, and at the same time the uniqueness of the initial behavior of the processes.

In conclusion we note the following. The result obtained can be heuristically found from the following argument. Physically it is obvious that the initial direction of the process coincides with that of a process in the case of contact of the halflines $x < 0$ and $x > 0$ at the moment $t = 0$, which are filled with heat-conducting material at critical temperatures equal to ϑ and 0, and initial temperatures 1 and 0 respectively. In this case the problem automatically has a solution of the form

$$u(x,t) = A_i + B_i \operatorname{erf}\left(\frac{x}{2a_i\sqrt{t}}\right); \quad z = -2\mu_1\sqrt{t}; \quad y = 2\mu_3\sqrt{t}, \tag{5.5.34}$$

where $i = 1$ for $-\infty < x < z(t)$, $i = 2$ for $z(t) < x < 0$, $i = 3$ for $0 < x < y(t)$ and $i = 4$ for $y(t) < x < \infty$, and also with $y(t) = 0$ in the case corresponding to process B and $z(t) = 0$ in the case C. It is easily seen that in case A

$$A_1 = 1 + B_1; \quad A_2 + A_3 = -B_3 \operatorname{erf}\left(\frac{\mu_3}{a_3}\right); \quad A_4 = B_4 = 0;$$

(5.5.35)
$$B_1 = -(1-\vartheta)\operatorname{erfc}^{-1}\left(\frac{\mu_1}{a_1}\right); \quad B_2 = \lambda_2 B_3;$$

$$B_3 = -\vartheta\left[\operatorname{erf}\left(\frac{\mu_3}{a_3}\right) + \lambda_2 \operatorname{erf}\left(\frac{\mu_1}{a_2}\right)\right]^{-1}.$$

In case B

$$A_1 = 1 + B_1; \quad A_2 = A_4 = B_2 = B_4 = 0;$$

(5.5.36)
$$B_1 = -(1-\vartheta)\operatorname{erfc}^{-1}\left(\frac{\mu_1}{a_1}\right).$$

and in case C

$$A_1 = 1 + B_1; \quad A_3 = A_1; \quad A_4 = B_4 = 0;$$

(5.5.37)
$$B_1 = \lambda_1 B_3;$$

$$B_3 = -\left(\lambda_1 + \operatorname{erf}\frac{\mu_3}{a_3}\right)^{-1}.$$

Inserting (5.5.34) into the Stefan conditions $(5.5.2_4)$, $(5.5.2_5)$, $(5.5.3_3)$ and $(5.5.4_4)$, we obtain the same results as were obtained more rigorously above.

Chapter VI

The Spatial Stefan Problem. Reduction to One Dimension. The Multi-phase Stationary Problem. A Filtration Analog to the Stefan Problem

§1. The Stefan problem with spherical symmetry [16]

The spatial Stefan problem in its general form is not yet completely solved. The existence of a classical solution has been established up to now only in two cases, each of which can be reduced to a one-dimensional problem, namely the Stefan problem with spherical or cylindrical symmetry. The problem of this type is in principle not different from the one-dimensional problem investigated in the preceding chapters. In addition, the analysis of two concrete problems having spherical symmetry (the problem of dissolution of a gas bubble in a liquid and the problem of evaporation of a spherical drop) is to be found in the papers of A. Friedman [41], [42], cited in the Introduction. In view of this, we limit ourselves to a brief description of an algorithm for the solution of a two-layer problem, analogous to the problem of crystallization of a melt upon insertion of a plate of more refractory material, which was investigated in the previous chapter.

Thus we examine the problem of determining $U(x,t)$ and $y(t)$ from the following conditions:

(6.1.1) $U(x,t)$ is uniformly bounded for $0 < x < y(t)$; $t > 0$;

(6.1.2) $$a^2(x)\left(\frac{\partial^2 U}{\partial x^2}+\frac{2}{x}\frac{\partial U}{\partial x}\right)=\frac{\partial U}{\partial t};\quad 0<x<y(t);\quad x\neq 1;\quad t>0;$$

(6.1.3) $$U|_{x=y(t)}=0;\quad t>0;\quad U|_{t=0}=1;\quad 0<x<1;$$

(6.1.4) $$U|_{x=1-0}=U|_{x=1+0};\quad k_1\frac{\partial U}{\partial t}\bigg|_{x=1-0}=k_2\frac{\partial U}{\partial x}\bigg|_{x=1+0};$$

(6.1.5) $$\dot{y}(t)=-\alpha+\beta\frac{\partial U}{\partial x}\bigg|_{x=y(t)};\quad y(0)=1.$$

Here, as in Chapter V,

$$(6.1.6)\quad a^2(x) = \begin{cases} a_1^2 = 1; & 0 < x < 1; \quad k_i = \text{const} > 0; \quad i = 1,2 \\ a_2^2 = a^2; & 1 < x < y(t); \quad \alpha = \text{const} > 0; \quad \beta = \text{const} < 0. \end{cases}$$

We set

$$(6.1.7)\quad xU(x,t) = \begin{cases} u_1(x,t), & 0 < x < 1; \\ u_2(x,t), & 1 < x < y(t); \end{cases}$$
$$u_1|_{x=1-0} = f(t); \quad \left.\frac{\partial u_2}{\partial x}\right|_{x=y(t)-0} = v(t) \equiv \frac{w(t)}{\sqrt{t}},$$

reducing problem (6.1.1)—(6.1.5) to a problem of determining u_i $(i = 1,2)$, f, v and y such that

$$(6.1.8_1)\quad \frac{\partial^2 u_1}{\partial x^2} = \frac{\partial u_1}{\partial t}; \quad 0 < x < 1; \quad t > 0;$$

$$(6.1.8_2)\quad a^2 \frac{\partial^2 u_2}{\partial x^2} = \frac{\partial u_2}{\partial t}; \quad 1 < x < y(t); \quad t > 0;$$

$$(6.1.8_3)\quad u_1|_{x=0} = 0, \quad t > 0; \quad u_1|_{t=0} = x, \quad 0 < x < 1; \quad u_2|_{x=y(t)} = 0, \quad t > 0;$$

$$(6.1.8_4)\quad u_1 = u_2 = f; \quad k_1\left(\frac{\partial u_1}{\partial x} - f\right) = k_2\left(\frac{\partial u_2}{\partial x} - f\right) \text{ for } x = 1, \quad t > 0;$$

$$(6.1.8_5)\quad \dot{y}(t) = -\alpha + \frac{\beta v(t)}{y(t)}; \quad y(0) = 1.$$

Let G_1 and G_2 be Green's functions for the equation $L_a u = 0$, defined in agreement with (0.2.8), (0.2.9), and let g_1 and g_2 be the Green's functions of the first and second boundary problems for the equation $Lu = 0$ on the interval $[0,1]$:

$$(6.1.9)\quad g_i(x,\xi,t) = \sum_{n=-\infty}^{+\infty} [E(x - \xi + 2n, t) + (-1)^i E(x + \xi - 2n, t)]$$

$(i = 1,2)$. In this case the solution u_1, u_2 of problem $(6.1.8_i)$ must be defined by the equations

$$(6.1.10_1)\quad u_1(x,t) = \int_0^1 \xi g_1(x,\xi,t)\,d\xi - \int_0^t f(\tau)\frac{\partial}{\partial \xi} g_1(x,1,t-\tau)\,d\tau;$$

$$(6.1.10_2)\quad u_2(x,t) = a^2 \int_0^t f(\tau)\frac{\partial}{\partial \xi} G_1(x,1,t-\tau)\,d\tau + a^2 \int_0^t \frac{w(\tau)}{\sqrt{\tau}} G_1(x,y(\tau),t-\tau)\,d\tau.$$

Differentiating $(6.1.10_i)$ $(i = 1, 2)$ with respect to x for $x \neq 1$, and correspondingly $x \neq 1$, $x \neq y(t)$, using the adjointness of G_1 and G_2, g_1 and g_2 and the obvious relation

$$\int_0^1 g_2(x, \xi, t)\, d\xi = 1, \tag{6.1.11}$$

we obtain

$$\frac{\partial}{\partial x} u_1 = 1 - g_2(x, 1, t) + \int_0^t f(\tau) \frac{\partial}{\partial t} g_2(x, 1, t - \tau)\, d\tau; \tag{6.1.12$_1$}$$

$$\begin{aligned} \frac{\partial}{\partial x} u_2 = &- \int_0^t f(\tau) \frac{\partial}{\partial t} G_2(x, 1, t - \tau)\, d\tau \\ &- a^2 \int_0^t \frac{w(\tau)}{\sqrt{\tau}} \frac{\partial}{\partial \xi} G_2(x, y(\tau), t - \tau)\, d\tau. \end{aligned} \tag{6.1.12$_2$}$$

Replacing t by s in $(6.1.12_i)$, integrating with respect to s from $\eta > 0$ to t, changing the order of integration, using the relations

$$\begin{aligned} &- a^2 \int_\tau^d \frac{\partial}{\partial \xi} G_2(x, y(\tau), s - \tau)\, ds \\ &\quad = \frac{1}{2}\left[\operatorname{erfc} \frac{y(t) - x}{2a\sqrt{t - \tau}} + \operatorname{erfc} \frac{x + y(\tau) - 2}{2a\sqrt{t - \tau}} \right], \end{aligned} \tag{6.1.13}$$

which are valid for $1 < x < y(\eta)$, and passing to the limit first with respect to x for $x \to 1$, and then with respect to η for $\eta \to 0$, we obtain

$$\begin{aligned} \int_0^t \frac{\partial u_1}{\partial x}\bigg|_{x=1} ds &= t - \int_0^t g_2(1, 1, s)\, ds \\ &\quad + \int_0^t f(\tau) g_2(1, 1, t - \tau)\, d\tau; \end{aligned} \tag{6.1.14$_1$}$$

$$\begin{aligned} \int_0^t \frac{\partial u_2}{\partial x}\bigg|_{x=1} ds &= - \int_0^t f(\tau)\, G_2(1, 1, t - \tau)\, d\tau \\ &\quad + \int_0^t \frac{w(\tau)}{\sqrt{\tau}} \operatorname{erfc} \frac{y(\tau) - 1}{2a\sqrt{t - \tau}}\, d\tau. \end{aligned} \tag{6.1.14$_2$}$$

On the other hand, condition $(6.1.8_4)$ gives

$$(k_1 - k_2) \int_0^t f(\tau)\, d\tau = k_1 \int_0^t \frac{\partial u_1}{\partial x}\bigg|_{x=1} ds - k_2 \int_0^t \frac{\partial u_2}{\partial x}\bigg|_{x=1} ds. \tag{6.1.14$_3$}$$

Comparing $(6.1.14_i)$ $(i = 1, 2, 3)$, we obtain

$$(k_1 - k_2)\int_0^t f(\tau)\,d\tau = k_1\left(t - \int_0^t g_2(1,1,t-\tau)\,d\tau\right)$$
$$+ \int_0^t f(\tau)K(t-\tau)\,d\tau - k_2\int_0^t \frac{w(\tau)}{\sqrt{\tau}}\operatorname{erfc}\frac{y(\tau)-1}{2a\sqrt{t-\tau}}\,d\tau. \tag{6.1.15}$$

Here

$$K(t) = k_1 g_2(1,1,t) + k_2 G_2(1,1,t)$$
$$\equiv \left(k_1 + \frac{k_2}{a}\right)\frac{1}{\sqrt{\pi t}} + 4k_1\sum_{n=1}^{\infty} E(2n,t). \tag{6.1.16}$$

From (6.1.16) it follows that (6.1.15) is a generalized equation of Abel. Its inversion gives

$$f(t) = \Phi(t) + (1+\delta)\int_0^t f(\tau)R(t-\tau)\,d\tau$$
$$+ a^2(1-\delta)\int_0^t w(\tau)E(y(\tau)-1, a^2(t-\tau))\frac{d\tau}{\sqrt{\tau}}, \tag{6.1.17}$$

where

$$\Phi(t) = \frac{1+\delta}{2} + (1+\delta)\sum_{n=1}^{\infty}\operatorname{erfc}\frac{n}{\sqrt{t}} - (1+\delta)\frac{\sqrt{t}}{\sqrt{\pi}};$$
$$R(t) = \frac{1-a\lambda}{2\sqrt{\pi t}} - \frac{2}{\sqrt{\pi}t^{3/2}}\sum_{n=1}^{\infty} n\exp\left(-\frac{n^2}{t}\right), \tag{6.1.18}$$

$$\lambda = \frac{k_2}{k_1}\cdot\frac{1}{a}; \qquad \delta = \frac{1-\lambda}{1+\lambda}. \tag{6.1.18*}$$

On the other hand, passing to the limit in $(6.1.12_2)$ for $x \to y(t)$ and using (6.1.7), we obtain

$$w(t) = -2\sqrt{t}\int_0^t f(\tau)\frac{\partial}{\partial t}G_2(y(t),1,t-\tau)\,d\tau$$
$$- 2a^2\int_0^t w(\tau)\sqrt{\frac{t}{\tau}}\frac{\partial}{\partial \xi}G_2(y(t),y(\tau),t-\tau)\,d\tau. \tag{6.1.19}$$

The system (6.1.17), (6.1.19) is complete upon adjoining condition $(6.1.8_5)$, which in integral form becomes

$$y(t) = 1 - \alpha t + \beta\int_0^t \frac{w(\tau)}{y(\tau)}\cdot\frac{d\tau}{\sqrt{\tau}}. \tag{6.1.20}$$

As in Chapter V, for the solution of this system it is necessary to examine the initial values $f(t)$ and $w(t)$. To this end, setting $\tau = tx$ in (6.1.17) and passing to the limit for $t \to 0$, we obtain

$$f(0) = \frac{1}{1+\lambda}\left[1 + \frac{\lambda a}{\sqrt{\pi}}\, w(0) \int_0^1 \frac{\exp\left(-\gamma^2 \frac{x}{1-x}\right)}{\sqrt{x}\sqrt{1-x}}\, dx\right] \tag{6.1.21}$$
$$= \frac{1}{1+\lambda}\left[1 + \lambda a w(0)\sqrt{\pi}\exp(\gamma^2)\operatorname{erfc}\gamma\right].$$

Similarly, from (6.1.19) we obtain

$$w(0) = \lim_{t\to 0} I_1 + \lim_{t\to 0} I_2, \tag{6.1.22}$$

where

$$I_1 = 2\int_0^1 f(tx)\sqrt{t}\frac{\partial}{\partial x} G_2(y(t), 1, t(1-x))\, dx;$$
$$I_2 = -2a^2 \int_0^1 w(tx)\frac{t}{\sqrt{x}}\frac{\partial}{\partial \xi} G_2(y(t), y(tx), t(1-x))\, dx. \tag{6.1.23}$$

Inserting the value $G_2(x, \xi, t)$ from (0.2.9) and making the calculations, we easily see that

$$\lim_{t\to 0} I_1 = -\frac{2f(0)}{a\sqrt{\pi}}\exp(-\gamma^2);$$
$$\lim_{t\to 0} I_2 = -w(0)\left[\operatorname{erf}\gamma - \operatorname{erfc}\gamma\right], \tag{6.1.24}$$

from which

$$w(0)\operatorname{erf}\gamma = -\frac{f(0)}{a\sqrt{\pi}}\exp(-\gamma^2). \tag{6.1.25}$$

Here

$$\gamma = \beta w(0)/a. \tag{6.1.26}$$

Eliminating $f(0)$ from (6.1.25) and (6.1.21), we finally obtain

$$w(0) = -\frac{\exp(-\gamma^2)}{\lambda a\sqrt{\pi}} - \frac{1}{\lambda} w(0)\operatorname{erf}\gamma. \tag{6.1.27}$$

Thus $w(0)$ in the problem with spherical symmetry coincides with the value of $w(0)$ in the one-dimensional problem, as is obvious a priori.

We will show the following. If we examine the problem (6.1.8$_i$) on the

interval $t > t_0 > 0$, replacing the initial condition $y(0) = 1$, $u|_{t=0} = 1$ by the condition

$$u_i|_{t=t_0} = \phi_i(x) \quad (i = 1, 2); \quad y(t_0) = l, \tag{6.1.28}$$

with ϕ_i satisfying the conditions of continuity in accordance with the boundary conditions, i.e.

$$\begin{aligned} &\phi_1(0) = 0; \quad \phi_1 = \phi_2; \\ &k_1(\dot{\phi}_1 - \phi_1) = k_2(\dot{\phi}_2 - \phi_2) \text{ for } x = 1; \quad \phi_2(l) = 0, \end{aligned} \tag{6.1.29}$$

then the system (6.1.17), (6.1.19), (6.1.20) is replaced by the system

$$\begin{aligned} f(t) &= \Phi^*(t) + (1+\delta)\int_{t_0}^{t} f(\tau) R(t-\tau)\, d\tau \\ &\quad + a^2(1-\delta)\int_{t_0}^{t} v(\tau) E(y(\tau) - 1, a^2(t-\tau))\, d\tau; \\ v(t) &= 2f(t_0) G_2(y(t), 1, t - t_0) \\ &\quad + 2\int_1^l \frac{\partial}{\partial \xi} u_2(\xi, t_0) G_2(y(t), \xi, t - t_0)\, d\xi \\ &\quad - 2\int_{t_0}^{t} f(\tau) \frac{\partial}{\partial t} G_2(y(t), 1, t-\tau)\, d\tau \\ &\quad - 2a^2 \int_{t_0}^{t} v(\tau) \frac{\partial}{\partial \xi} G_2(y(t), y(\tau), t-\tau)\, d\tau; \\ y(t) &= l + \int_{t_0}^{t} [\beta v(\tau) - \alpha]\, d\tau. \end{aligned} \tag{6.1.30}$$

Here

$$\begin{aligned} \Phi^*(t) &= (1-\delta)\int_1^l u_2(\xi, t_0) E(\xi - 1, a^2(t - t_0))\, d\xi \\ &\quad + (1+\delta)\int_0^1 u_1(\xi, t_0) \Big\{ E(1 - \xi, t - t_0) \\ &\quad + \sum_{n=1}^{\infty} [E(1 + 2n - \xi, t - t_0) \\ &\quad - E(2n - 1 + \xi, t - t_0)] \Big\}\, d\xi. \end{aligned} \tag{6.1.31}$$

To the system (6.1.30) we apply the theory developed in Chapter I.

Further, it is easily proved that this system is equivalent to the original problem in the sense of Chapters I and II. Thus in the case of a two-phased initial state with continuously agreeing boundary and initial data, the problem with spherical symmetry automatically has a solution in the small. Using the method developed in Chapters II and V, it is obviously possible to prove the extendability of the solution to this problem, to any time t, as long as $y(t)$ does not become equal to unity, i.e. to prove the existence of the solution in the large.

In addition, it is possible to prove that if $y(t)$ is known, then problem $(6.1.8_1)$—$(6.1.8_4)$ is automatically solvable.

This consideration permits the construction of an algorithm for solving the problem $(6.1.8_1)$—$(6.1.8_5)$ which is exactly analogous to the algorithm for the solution of problem (5.1.1)—(5.1.4).

At present we limit ourselves to this brief remark. As in the case of a one-dimensional problem, a more detailed description of the algorithm, together with numerical examples of its application, is presented in Supplement 3, §2.

§2. The Stefan problem with cylindrical symmetry [128]

We now examine the Stefan problem with cylindrical symmetry. For simplicity we limit ourselves to a two-phase problem with nondegenerate initial conditions and continuous agreement of boundary and initial data. Thus we examine the problem

$$(6.2.1_1)\qquad \frac{\partial^2 u_1}{\partial x^2}+\frac{1}{x}\frac{\partial u_1}{\partial x}=\frac{\partial u_1}{\partial t}\ \text{for}\ y(t)<x<1;\quad t>0;$$

$$(6.2.1_2)\qquad \frac{\partial^2 u_2}{\partial x^2}+\frac{1}{x}\frac{\partial u_2}{\partial x}=\frac{1}{a^2}\frac{\partial u_2}{\partial t}\ \text{for}\ 0<x<y(t);\quad t>0;$$

$$(6.2.1_3)\ u_1(1,t)=f(t);\quad u_i(x,0)=\phi_i(x);\quad u_i(y(t),t)=0;\quad t>0\ \ (i=1,2);$$

$$(6.2.1_4)\qquad \dot{y}(t)=\frac{\partial}{\partial x}u_1(y(t),t)-\frac{\partial}{\partial x}u_2(y(t),t);\quad y(0)=l\in(0,1).$$

We will assume that

$$(6.2.1_5)\quad f(t)\leqq 0;\ \ \phi_1(x)\leqq 0;\ \ \phi_2(x)\geqq 0;\ \ \phi_1(l)=\phi_2(l)=0;\ \ f(0)=\phi_1(1),$$

with $f(t)$ and $\phi_i(x)$ having a sufficient number of derivatives. Set

$$(6.2.2)\qquad v_0(t)=\frac{\partial}{\partial x}u_1(1,t);\quad v_i(t)=\frac{\partial}{\partial x}u_i(y(t),t)\quad (i=1,2)$$

and

$$E_n(x,\xi,a^2(t-\tau)) = \frac{\exp\left(-\dfrac{x^2+\xi^2}{4a^2(t-\tau)}\right)}{2a^2(t-\tau)}\, I_n\left(\frac{x\xi}{2a^2(t-\tau)}\right), \tag{6.2.3}$$

where $I_n(z)$ is a Bessel function of minimum argument of order n. We recall that $E_0(x,\xi,a^2(t-\tau))$ is an instantaneous cylindrical heat source.

From conditions $(6.2.1_1)$—$(6.2.1_4)$ and the definitions (6.2.2) and (6.2.3) it follows that $u_i(x,t)$ is determined from the relations

$$\begin{aligned} u_1(x,t) &= \int_0^t v_0(\tau)\,E_0(x,1,t-\tau)\,d\tau - \int_0^t f(\tau)\,\frac{\partial}{\partial\xi}\,E_0(x,1,t-\tau)\,d\tau \\ &\quad + \int_l^1 \xi\phi_1(\xi)\,E_0(x,\xi,t)\,d\xi - \int_0^t v_1(\tau)\,y(\tau)\,E_0(x,y(\tau),t-\tau)\,d\tau; \end{aligned} \tag{6.2.4$_1$}$$

$$\begin{aligned} u_2(x,t) &= \int_0^l \xi\phi_2(\xi)\,E_0(x,\xi,a^2t)\,d\xi \\ &\quad + a^2\int_0^t v_2(\tau)\,y(\tau)\,E_0(x,y(\tau),a^2(t-\tau))\,d\tau. \end{aligned} \tag{6.2.4$_2$}$$

Let $y(t) < x < 1$ and $t > 0$. Differentiating $(6.2.4_1)$ with respect to x, we find that

$$\begin{aligned} \frac{\partial u_1}{\partial x} &= [f(0) - \phi_1(1)]E_1(x,1,t) + l\phi_1(l)\,E_1(x,l,t) \\ &\quad + \int_l^1 \dot{\phi}_1(\xi)\,\xi E_1(x,\xi,t)\,d\xi + \int_0^t \dot{f}(\tau)\,E_1(x,1,t-\tau)\,d\tau \\ &\quad + \int_0^t v_0(\tau)\,\frac{\partial}{\partial x}\,E_0(x,1,t-\tau)\,d\tau \\ &\quad - \int_0^t v_1(\tau)\,y(\tau)\,\frac{\partial}{\partial x}\,E_0(x,y(\tau),t-\tau)\,d\tau. \end{aligned} \tag{6.2.5}$$

This follows from the equations

$$\begin{aligned} &\frac{\partial E_0}{\partial x} = -\frac{\partial E_1}{\partial\xi} - \frac{E_1}{\xi} = -\frac{x}{2a^2(t-\tau)}\,E_0 + \frac{\xi}{2a^2(t-\tau)}\,E_1; \\ &a^2\frac{\partial^2 E_0}{\partial x\,\partial\xi} = -\frac{\partial E_1}{\partial t} = \frac{x\xi}{2a^2(t-\tau)}\,E_0 - \frac{x^2+\xi^2}{4a^2(t-\tau)^2}\,E_1; \\ &a^2\left(\frac{\partial^2 E_0}{\partial x^2} + \frac{1}{x}\,\frac{\partial E_0}{\partial x}\right) = \frac{\partial E_0}{\partial t}; \quad E_i(x,\xi,t-\tau) = E_i(\xi,x,t-\tau), \end{aligned} \tag{6.2.6}$$

which are easily confirmed directly. Using the conjugate condition in $(0.2.1_5)$ and passing to the limit for $x \to 1$ with the aid of the gap theorem for the heat potential, we obtain

$$
\begin{aligned}
v_0(t) &= 2\int_0^t v_0(\tau)\frac{\partial}{\partial x}E_0(1,1,t-\tau)\,d\tau \\
&\quad + 2\int_0^t \dot{f}(\tau)E_1(1,1,t-\tau)\,d\tau + 2\int_l^1 \dot{\phi}_1(\xi)\xi E_1(1,\xi,t)\,d\xi \\
&\quad - 2\int_0^t y(\tau)v_1(\tau)\frac{\partial}{\partial x}E_0(1,y(\tau),t-\tau)\,d\tau.
\end{aligned} \tag{6.2.7$_1$}
$$

Similarly,

$$
\begin{aligned}
v_1(t) &= 2\int_0^t v_0(\tau)\frac{\partial}{\partial x}E_0(y(t),1,t-\tau)\,d\tau \\
&\quad + 2\int_0^t \dot{f}(\tau)E_1(y(t),1,t-\tau)\,d\tau + 2\int_l^1 \dot{\phi}_1(\xi)\xi E_1(y(t),\xi,t)\,d\xi \\
&\quad - 2\int_0^t y(\tau)v_1(\tau)\frac{\partial}{\partial x}E_0(y(t),y(\tau),t-\tau)\,d\tau.
\end{aligned} \tag{6.2.7$_2$}
$$

and

$$
\begin{aligned}
v_2(t) &= 2\int_0^l \dot{\phi}_2(\xi)\xi E_1(y(t),\xi,a^2t)\,d\xi \\
&\quad + 2a^2\int_0^t y(\tau)v_2(\tau)\frac{\partial}{\partial x}E_0(y(t),y(\tau),a^2(t-\tau))\,d\tau.
\end{aligned} \tag{6.2.7$_3$}
$$

The system $(6.2.7_i)$ is completed by adjoining the Stefan condition

$$
y(t) = l + \int_0^t [v_1(\tau) - v_2(\tau)]\,d\tau. \tag{6.2.7$_4$}
$$

Thus, just as in the one-dimensional case, the problem reduces to a system of functional equations of Volterra type, sufficing for the determination of the boundary between phases and the gradient of "temperature" of each phase at the boundary of its domain of existence. We note that by introducing the Green's function of the first boundary problem for a cylinder it is possible to exclude $v_0(t)$ from investigation. However, this rather complicates the subsequent analysis, and we therefore commonly use the extended system $(6.2.7_i)$ of four equations.

The system $(6.2.7_i)$ is solved by iteration, setting

$$
\begin{aligned}
&v_{in}(t) = V_i(t\,|\,y_{n-1}, v_{j,n-1});\\
&y_n(t) = l + \int_0^t (v_{1n}(\tau) - v_{2n}(\tau))\,d\tau \quad (n = 0, 1, 2\cdots).
\end{aligned} \tag{6.2.8}
$$

Here $V_i(t, y, v_j)$ denotes the right side of $(6.2.7_i)$ $(i = 1, 2, 3)$.

In order to establish the convergence of the iteration process we note the following. As is known [54], $I_n(z)$ can be written in the form

$$
\begin{aligned}
I_n(z) = {} & \frac{\exp z}{\sqrt{2\pi z}}\,[1 + R_1^n(z\exp(i\pi))]\\
& + \frac{\exp(-z - i\pi(n + 1/2))}{\sqrt{2\pi z}}\,(1 + R_1^n(z));\\
& \left(n = 0, 1, 2, \cdots;\ -\frac{3}{2}\,\pi < \arg z < \frac{\pi}{2}\right).
\end{aligned} \tag{6.2.9}
$$

Here

$$
R_1^n(z) = \frac{1}{2z\Gamma\left(n - \frac{1}{2}\right)} \int_0^\infty \exp(-\zeta)\cdot\zeta^{n+1/2}d\zeta \int_0^1 \left(1 + \frac{\zeta t}{2z}\right)^{n-3/2} dt, \tag{6.2.10}
$$

where moreover the integration is carried out along an arbitrary line

$$
\zeta = \lambda\exp(i\psi);\quad -\frac{\pi}{2} < \psi < \frac{\pi}{2}. \tag{6.2.11}
$$

For $-3\pi/2 + \epsilon \leqq \arg z \leqq \pi/2$ the sequences $R_1^n(z)$, $R_1^n(z\exp(i\pi))$ and their successive derivatives are uniformly convergent to integrals of holomorphic functions. Hence

$$
\left|\frac{d^m R_1^n(z)}{dz^n}\right| \leqq \frac{C_m}{|z^{m+1}|};\quad \left|\frac{d^m R_1^n(z\exp(i\pi))}{dz^n}\right| \leqq \frac{C_m}{|z^{m+1}|}, \tag{6.2.12}
$$

where C_m is a constant independent of z.

From (6.2.9) and (6.2.3) it follows that $E_n(x, \xi, t)$ and $\partial E_0(x, \xi, t)/\partial x$ are representable in the form

$$
\begin{aligned}
E_n(x, \xi, t) = {} & \frac{\exp\left(-\frac{(x-\xi)^2}{4t}\right)}{2\sqrt{\pi x\xi t}}\left[1 + R_1^n\left(\frac{x\xi\exp(i\pi)}{2t}\right)\right]\\
& + \frac{\exp\left[-\frac{(x+\xi)^2}{4t} - i\pi\left(n + \frac{1}{2}\right)\right]}{2\sqrt{\pi x\xi t}}\left[1 + R_1^n\left(\frac{x\xi}{2t}\right)\right];
\end{aligned} \tag{6.2.13}
$$

$$
\begin{aligned}
\frac{\partial}{\partial x} E_0(x,\xi,t) = &-\frac{(x-\xi)\exp\left(-\dfrac{(x-\xi)^2}{4t}\right)}{4\sqrt{\pi x\xi t^3/2}}\left[1+R_1^0\left(\frac{x\xi\exp(i\pi)}{2t}\right)\right] \\
&-\frac{(x+\xi)\exp\left[-\dfrac{(x+\xi)^2}{4t}-i\dfrac{\pi}{2}\right]}{4\sqrt{\pi x\xi t^{3/2}}}\left[1+R_1^0\left(\frac{x\xi}{2t}\right)\right] \\
&+\frac{\sqrt{\xi}}{4\sqrt{\pi x}}\frac{\exp\left(-\dfrac{(x-\xi)^2}{4t}+i\pi\right)}{t^{3/2}}\dot{R}_1^0\left(\frac{x\xi\exp(i\pi)}{2t}\right) \\
&+\frac{\sqrt{\xi}}{4\sqrt{\pi x}}\frac{\exp\left(-\dfrac{(x+\xi)^2}{4t}-i\dfrac{\pi}{2}\right)}{t^{3/2}}\dot{R}_1^0\left(\frac{x\xi}{2t}\right) \\
&-\frac{\exp\left(-\dfrac{(x-\xi)^2}{4t}\right)}{4\sqrt{\pi x^3\xi t}}\left[1+R_1^0\left(\frac{x\xi\exp(i\pi)}{2t}\right)\right] \\
&-\frac{\exp\left(-\dfrac{(x+\xi)^2}{4t}-i\dfrac{\pi}{2}\right)}{4\sqrt{\pi x^3\xi t}}\left[1+R_1^0\left(\frac{x\xi}{2t}\right)\right].
\end{aligned}
\tag{6.2.14}
$$

We set

$$
\begin{aligned}
&\delta E_n = E_n(x_2(t),y_2(\tau),t-\tau) - E_n(x_1(t),y_1(\tau),t-\tau); \\
&\delta\frac{\partial E_0}{\partial x} = \frac{\partial}{\partial x}E_0(x_2(t),y_2(\tau),t-\tau) - \frac{\partial}{\partial x}E_0(x_1(t),y_1(\tau),t-\tau); \\
&\delta x = x_2(t)-x_1(t); \quad \delta y = y_2(\tau)-y_1(\tau); \\
&\delta xy = x_2(t)\,y_2(\tau) - x_1(t)\,y_1(\tau); \\
&x(t) = x_1(t)+\Theta\delta x; \quad y(\tau) = y_1(\tau)+\Theta\delta y; \quad 0<\Theta<1.
\end{aligned}
\tag{6.2.15}
$$

Using the formulas with finite increments and (6.2.12), we find that

$$
|\delta E_n| < N\left\{\frac{|x(t)-y(\tau)|}{t-\tau}\cdot\frac{|\delta(x(t)-y(\tau))|}{\sqrt{t-\tau}}\exp\left[-\frac{(x(t)-y(\tau))^2}{4(t-\tau)}\right] + \frac{\delta(x(t)\,y(\tau))}{\sqrt{t-\tau}}\right\};
\tag{6.2.16}
$$

$$\left|\delta\frac{\partial E_0}{\partial x}\right| < N\left\{\frac{|\delta(x(t)-y(\tau))|}{(t-\tau)^{3/2}}\exp\left[-\frac{(x(t)-y(\tau))^2}{4(t-\tau)}\right]\right.$$

$$+\frac{(x(t)-y(\tau))^2}{(t-\tau)^{5/2}}|\delta(x(t)-y(\tau))|\exp\left[-\frac{(x(t)-y(\tau))^2}{4(t-\tau)}\right] \tag{6.2.17}$$

$$\left.+\frac{|\delta x(t)|+|\delta y(\tau)|}{\sqrt{t-\tau}}\right\}$$

for $0<\rho\leqq\min\min\{x_i(t),y_i(\tau),1-x_i(t),1-y_i(\tau)\}$. Here N depends only on C_m $(m=0,1,2)$ and on ρ, not on T.

The estimates thus obtained easily allow us to establish the applicability to the system (6.2.7$_i$) $(i=1,2,3,4)$ of the principle of contraction mappings, and together with it, the convergence of the Picard iteration process to a unique solution. Simultaneously, as in Chapter I, we establish the stability of $v_i(t)$ and $\dot{y}(t)$ relative to small perturbations of the boundary and initial data, on the one hand, and the parameters characterizing the properties of each phase, on the other. Finally, the equivalence of the solution of this system to the solution of the original problem in the sense of Theorem 5 is proved in exactly the same way as this theorem. Namely, the problem reduces to the establishment of a unique solution to the system of integral equations

$$w_0(t)=2\int_0^t w_0(\tau)\frac{\partial}{\partial\xi}E_0|_{\xi=x=1}d\tau$$

$$-2\int_0^t w_1(\tau)\left(\frac{\partial}{\partial\xi}-\dot{\xi}\right)E_0\cdot\xi|_{\xi=y(\tau);x=1}d\tau;$$

$$w_1(t)=2\int_0^t w_0(\tau)\frac{\partial}{\partial\xi}E_0|_{\xi=1;x=y(t)}d\tau \tag{6.2.18}$$

$$-2\int_0^t w_1(\tau)\left(\frac{\partial}{\partial\xi}-\dot{\xi}\right)E_0\xi|_{\xi=y(\tau);x=y(t)}d\tau;$$

$$w_2(t)=2\int_0^t w_2(\tau)\left(a^2\frac{\partial}{\partial\xi}-\dot{\xi}\right)E_0\xi|_{\xi=y(\tau);x=y(t)}d\tau.$$

In the case of interest to us, $0<l=y(0)<1$, the uniqueness of the solution to this system is an obvious consequence of (6.2.16) and (6.2.17).

§3. The multi-phase stationary Stefan problem [140]

Although the nonstationary Stefan problem is essentially nonlinear and therefore in principle different from the usual linear boundary problems posed for the equation of heat conduction on a domain with preassigned

boundary, the corresponding stationary problem can be reduced in a trivial way to the usual linear boundary problem. In fact, let there be given a simply connected domain D^* bounded by a surface Σ_0, and an open set G_n contained in D^* together with its boundary Σ_n. We assume that every point of Σ_0 and Σ_n is regular relative to the Dirichlet problem.

Consider known continuous functions $f_0(p)$, $p \in \Sigma_0$, and $f_n(p)$, $p \in \Sigma_n$, such that

$$\min_{p\in\Sigma_n} f_n(p) = M > \max_{p\in\Sigma_0} f_0(p) = m \tag{6.3.1}$$

and constants $k_i > 0$ $(i = 0, \cdots, n-1)$ and T_i $(i = 1, \cdots, n-1)$ such that

$$m < T_1 < T_2 < \cdots < T_{n-1} < M. \tag{6.3.2}$$

We assume that $D = D^* \setminus \overline{G}_n$ and we examine the system of open sets G_i satisfying the conditions

a)

$$G_0 = D; \quad G_i \supset \overline{G}_{i+1} \quad (i = 0, 1, \cdots, n-1); \tag{6.3.3}$$

b) If Σ_{i+1} is a common part of the boundaries of $G_i \setminus G_{i+1}$ and G_{i+1}:

$$\Sigma_{i+1} = (G_i \setminus G_{i+1}) \cap \overline{G}_{i+1} \quad (i = 0, 1, 2, \cdots, n-2), \tag{6.3.4}$$

then Σ_{i+1} is the union of a finite number of smooth nonintersecting surfaces.

We pose the problem of finding $u(P)$ and G_i $(i = 1, \cdots, n-1)$ such that $u(p)$ is continuous in $\overline{D}$, harmonic in each of the domains $G_i \setminus \overline{G}_{i+1}$ and satisfies the conditions

$$u(p) = f_0(p), \quad p \in \Sigma_0; \quad u(p) = f_n(p), \quad p \in \Sigma_n; \tag{6.3.5}$$

$$u(p) = T_i, \quad p \in \Sigma_i \quad (i = 1, 2, \cdots, n-1); \tag{6.3.6}$$

$$k_i \frac{\partial u_i}{\partial n} = k_{i+1} \frac{\partial u_{i+1}}{\partial n}, \quad p \in \Sigma_{i+1} \quad (i = 0, 1, 2, \cdots, n-2). \tag{6.3.7}$$

Here u_i is the value of u in $G_i \setminus \overline{G}_{i+1}$ and n is the direction of the normal to Σ_{i+1} into the interior relative to G_{i+1}. We note that from these conditions, from (6.3.2) and from the maximum principle it follows that

$$\begin{aligned} &f_0(p) < u_0 < T_1; \cdots; \quad T_m < u_m < T_{m+1} \quad (m = 1, 2, \cdots, n-2); \\ &T_{n-1} < u_{n-1} < f_n(p). \end{aligned} \tag{6.3.8}$$

Let us assume that the solution $u(p)$ of the posed problem, which will be called the Dirichlet-Stefan problem, exists. We examine the sequences

$$
\begin{aligned}
&T_i^0 = T_i \quad (i = 1, 2, \cdots, n-1); \quad f_i^0(p) = f_i(p) \quad (i = 0, n);\\
&T_i^{m+1} = k_{m+1}(T_i^m - T_{m+1}^m) \quad (i = m+1, m+2, \cdots, n-1);\\
&f_0^{m+1} = k_m(f_0^m - T_{m+1}^m); \quad f_n^{m+1} = k_{m+1}(f_n^m - T_{m+1}^m)\\
&\qquad\qquad (m = 0, 1, \cdots, n-2).
\end{aligned} \tag{6.3.9}
$$

We set

$$
\begin{aligned}
&u^0(p) = u(p); \quad p \in D;\\
&u^{m+1}(p) = \begin{cases} k_m(u^m(p) - T_{m+1}^m); & p \in D \setminus \overline{G}_{m+1} \\ k_{m+1}(u^m(p) - T_{m+1}^m); & p \in G_{m+1} \end{cases}\\
&\qquad\qquad (m = 0, 1, \cdots, n-2).
\end{aligned} \tag{6.3.10}
$$

Obviously $u^{n-1}(p)$ is a solution to the Dirichlet problem

$$
\begin{aligned}
&\Delta u^{n-1}(p) = 0; \quad p \in D;\\
&u^{n-1}(p) = f_0^{n-1}(p); \quad p \in \Sigma_0;\\
&u^{n-1}(p) = f_n^{n-1}(p); \quad p \in \Sigma_n.
\end{aligned} \tag{6.3.11}
$$

In fact, from (6.3.5)—(6.3.9) it follows that $u^{n-1}(p)$ is harmonic in each of the domains $G_i \setminus \overline{G}_{i+1}$, continuous in $\overline{D}$ and has continuous normal derivatives on each of the boundaries Σ_i $(i = 1, \cdots, n-1)$. Hence from the theorem of continuation of harmonic functions it follows that u^{n-1} is harmonic everywhere in D. Finally, from the equations

$$
\begin{aligned}
&u^{n-1}(p) = f_0^{n-1}(p); \quad p \in \Sigma_0;\\
&u^{n-1}(p) = f_n^{n-1}(p); \quad p \in \Sigma_n,
\end{aligned} \tag{6.3.12}
$$

satisfied by construction, it follows that $u^{n-1}(p)$ satisfies the boundary conditions of problem (6.3.11), which proves our assertion. Together with this the uniqueness of the solution of the Dirichlet-Stefan problem is seen to hold, since the solution of the Dirichlet problem is automatically unique.

Conversely, let $u^{n-1}(p)$ be a solution of the Dirichlet problem (6.3.11). It automatically exists, since by the assumptions each point of the boundaries Σ_0 and Σ_n is regular. Since by construction the functions f_0^{n-1} and f_n^{n-1} have opposite signs, there exists an "iso"-surface Σ_{n-1} (not necessarily connected), on which

$$
u^{n-1}(p) = 0; \quad p \in \Sigma_{n-1}. \tag{6.3.13}
$$

This surface bounds a domain G_{n-1} within which

$$0 < u^{n-1}(p) < f_n^{n-1}(p); \quad p \in G_{n-1}. \tag{6.3.14}$$

At the same time

$$f_0^{n-1}(p) < u^{n-1}(p) < 0 \text{ for } p \in D \setminus \overline{G}_{n-1}. \tag{6.3.15}$$

Finally, from the fact that u^{n-1} is harmonic in D we obtain the continuity of $\partial u/\partial n$ on Σ_{n-1}.

We obtain the function u^{n-2} from u^{n-1} by using equations (6.3.10) and (6.3.9). We find that u^{n-2} is continuous in D, harmonic in each of the domains G_{n-1} and $D \setminus G_{n-1}$, and satisfies the boundary conditions

$$u^{n-2} = f_0^{n-2}, \quad p \in \Sigma_0; \quad u^{n-2} = f_n^{n-2}, \quad p \in \Sigma_n \tag{6.3.16}$$

and the conjugacy conditions

$$u^{n-2} = T_{n-1}^{n-2} \text{ on } \Sigma_{n-1}; \quad k_{n-2}\frac{\partial u^{n-2}}{\partial n_{i-0}} = k_{n-1}\frac{\partial u^{n-2}}{\partial n_{i+0}} \text{ on } \Sigma_{n-1}. \tag{6.3.17}$$

Here the index of the sign of the derivative denotes that of the normal corresponding to the limiting value outwards from the domains $D \setminus G_{n-1}$ and G_{n-1} respectively. If we continue the slope up to $u^0 = 0$, we can prove by induction that u is a solution to the original Dirichlet-Stefan problem.

Thus the equivalence of the Dirichlet problem and the Dirichlet-Stefan problem is completely proved.

In the more general case of quasilinear equations of elliptic type with piecewise continuous coefficients, the equivalence of the problem of Dirichlet type and that of Dirichlet-Stefan type already does not hold. The investigation of the stationary Stefan problem for such equations is the subject of the article [110], to which we refer the reader.

§4. Filtration analog of the Stefan problem [138]

Returning to the one-dimensional problem, we give a brief indication of the application of the method of reduction to integral equations to the solution of the filtration analog of the Stefan problem. We recall that the singularity of this problem occurs in the boundary condition of conjugation, and is coincident with the usual conjugacy conditions in the multi-layer thermal problem.

As a characteristic example we will investigate the problem of the

advance of a water-oil contact in an elastic layer (see Part 1, Chapter III, §2) under the assumption of nondegeneration at the initial moment of each of the domains entering in its conditions, and continuous agreement of boundary and initial data. In this case the solution to the problem consists of the determination of functions $v(x,t)$ and $y(t)$ such that

$$(6.4.1_1)\qquad \alpha^2(x)\frac{\partial^2 v}{\partial x^2}=\frac{\partial v}{\partial t};\quad -\infty<x<0;\quad x\neq -1;\quad x\neq y(t);\quad t>0;$$

$$(6.4.1_2)\qquad v(x,t) \text{ and } k(x)\frac{\partial v}{\partial x} \text{ are continuous for } x=-1 \text{ and } x=y(t);\ t>0;$$

$$(6.4.1_3)\qquad v(x,0)=\phi(x);\quad -\infty<x<0;\quad v(0,t)=\psi(t);\quad t>0;$$

$$(6.4.1_4)\qquad \dot{y}(t)=-\gamma\frac{\partial v}{\partial x}\Big|_{x=y(t)-0};\quad t>0;\quad y(0)=-l\in(-1,0).$$

Here

$$(6.4.1_5)\qquad \alpha(x)=\begin{cases}\alpha_1=1; & -\infty<x<-1\\ \alpha_2; & -1<x<y(t)\\ \alpha_3; & y(t)<x<0;\end{cases}$$

$$k(x)=\begin{cases}k_1; & -\infty<x<-1;\quad k_i=\text{const.}>0\\ k_2; & -1<x<y(t);\quad \alpha_i=\text{const.}>0\\ k_3; & y(t)<x<=0\quad (i=1,2,3).\end{cases}$$

We will assume that

$$(6.4.2)\qquad \phi(0)=\psi(0),$$

where ϕ and ψ are continuous for $-\infty<x\leqq 0$ and $t\geqq 0$ respectively, and everywhere have such derivatives as are needed for the construction of the solution. We assume that

$$(6.4.3)\qquad \begin{aligned}&\xi=x-y(t);\quad \tau=t;\quad z(\tau)=-y(t);\quad u(\xi,\tau)=v(x,t);\\ &\beta(\xi)=\alpha(x);\quad \kappa(\xi)=k(x);\quad \Phi(\xi)=\phi(x).\end{aligned}$$

Problem $(6.4.1_i)$ leads to the problem of determining $u(\xi,\tau)$ and $z(\tau)$ such that

$$(6.4.4_1)\qquad \beta^2(\xi)\frac{\partial^2 u}{\partial\xi^2}=\frac{\partial u}{\partial\tau}+\xi\dot{z}(\tau)\frac{\partial u}{\partial\xi};\quad -\infty<\xi<z(\tau);$$

$$\xi\neq z(\tau)-1;\quad \xi\neq 0;\quad \tau>0,$$

(6.4.4$_2$) $u(\xi,\tau)$ and $\kappa(\xi)\dfrac{\partial u}{\partial \xi}$ are continuous for $\xi = z(\tau) - 1, \xi = 0$ and $\tau > 0$;

(6.4.4$_3$) $$u(\xi,0) = \Phi(\xi); \quad -\infty < \xi < l; \quad u(z(\tau),\tau) = \psi(\tau); \quad \tau > 0;$$

(6.4.4$_4$) $$\dot{z}(\tau) = \gamma \frac{\partial u}{\partial \xi} \text{ for } \xi = z(\tau); \quad \tau > 0; \quad z(0) = l.$$

The coordinates (x,t) and (ξ,τ) will both be used.

We set

$$v|_{x=-1} = f_1(t); \quad v|_{x=y(t)} = f_2(t); \quad \frac{\partial v}{\partial x} = w(x,t); \quad \frac{\partial u}{\partial \xi} = q(\xi,\tau);$$

(6.4.5) $$w|_{x=-1+0} = F_1(t); \quad w|_{x=y(t)-0} = F_2(t); \quad w|_{x=-0} = F_3(t);$$

$$\lambda_i = \frac{k_{i+1}}{k_i} \quad (i = 1,2).$$

We then obtain

(6.4.6) $$w|_{x=-1-0} = \lambda_1 F_1; \quad w|_{x=y(t)+0} = \lambda_2 F_2;$$
$$q|_{\xi=z(\tau)-1-0} = \lambda_1 F_1; \quad q|_{\xi=+0} = \lambda_2 F_2; \quad q|_{\xi=z(\tau)-0} = F_3.$$

We seek a solution of the problem such that

(6.4.7) $$f_1(0) = \phi(-1); \quad f_2(0) = \phi(-l).$$

Let $G_1(x,\xi,t)$ be the fundamental solution of the heat equation for a two-layered space with a discontinuity for the parameter value $x = -1$, i.e. the solution of the problem

(6.4.8) $$\begin{cases} \alpha^2(x)\dfrac{\partial^2 G_1}{\partial x^2} = \dfrac{\partial G_1}{\partial t}; \quad -\infty < x < \infty; \quad x \neq 1; \quad t > 0; \\ G_1 \text{ and } k(x)\dfrac{\partial G_1}{\partial x} \text{ are continuous for } x = -1 \text{ and } t > 0; \end{cases}$$
$$G_1(x,\xi,0) = \delta(x-\xi).$$

Here $\delta(x)$ is the Dirac delta function, and

(6.4.9) $$\alpha(x) = \begin{cases} \alpha_1, & x < -1, \\ \alpha_2, & x > -1; \end{cases} \quad k(x) = \begin{cases} k_1, & x < -1, \\ k_2, & x > -1. \end{cases}$$

Moreover, let $G_1^*(x,\xi,t)$ be conjugate to $G_1(x,\xi,t)$. The expressions of

G_1 and G_1^* are obtained from equations (5.5.9) and (5.5.10) by replacing a_i by α_i and 1 by -1. Further, we define $G_0(x,\xi,t)$ and $G_0^*(x,\xi,t)$ as the fundamental solution and its conjugate, obtained from G_1 and G_1^* by replacing α_i and k_i by α_{i+1} and k_{i+1}, respectively, and shifting the point $x=-1$ to the point $x=0$. Finally let $g_1(x,\xi,t)$ and $g_2(x,\xi,t)$ be the Green's functions of the first and second boundary problems for the equation $\alpha_3^2 u_{xx}=u_t$ on the halfline $x<0$. We then have

$$v(x,t)=V(x,t|f_2,F_2,y), \tag{6.4.10}$$

where

$$\begin{aligned} V=&\int_{-l}^{0}\phi(\xi)g_1(x,\xi,t)\,d\xi-\lambda_2\alpha_3^2\int_0^t F_2(\tau)g_1(x,y(\tau),t-\tau)\,d\tau\\ &+\int_0^t f_2(\tau)\left(\alpha_3^2\frac{\partial}{\partial\xi}-\dot{y}(\tau)\right)g_1(x,y(\tau),t-\tau)\,d\tau\\ &-\alpha_3^2\int_0^t\psi(\tau)\frac{\partial}{\partial\xi}\,g_1(x,0,t-\tau)\,d\tau \end{aligned} \tag{6.4.11}$$

for $y(t)<x<0$, and

$$\begin{aligned} V=&\int_{-\infty}^{-l}\phi(\xi)\,G_1(x,\xi,t)\,d\xi+\alpha_2^2\int_0^t F_2(\tau)\,G_1(x,y(\tau),t-\tau)\,d\tau\\ &-\int_0^t f_2(\tau)\left(\alpha_2^2\frac{\partial}{\partial\xi}-\dot{y}(\tau)\right)G_1(x,y(\tau),t-\tau)\,d\tau \end{aligned} \tag{6.4.12}$$

for $-\infty<x<y(t)$.

Passing to the limit in (6.4.10) for $x\to-1-0$, we obtain

$$\begin{aligned} f_1(t)=&\int_{-\infty}^{-l}\phi(\xi)\,G_1(-1-0,\xi,t)\,d\xi\\ &+\alpha_2^2\int_0^t F_2(\tau)\,G_1(-1-0,y(\tau),t-\tau)\,d\tau\\ &-\int_0^t f_2(\tau)\left(\alpha_2^2\frac{\partial}{\partial\xi}-\dot{y}(\tau)\right)G_1(-1-0,y(\tau),t-\tau)\,d\tau. \end{aligned} \tag{6.4.13}$$

Differentiating (6.4.13), using (6.4.1$_1$), carrying out integration by parts, using the continuity of $\phi(\xi)$ and $\alpha^2(\xi)\,\partial G_1(x,\xi,t)/\partial\xi$ and $\alpha(\xi)k^*(\xi)\,G_1(x,\xi,t)$

for $\xi = -1$[1] we obtain

$$f_1(t) = R_1(t \mid f_2, F_2, y), \tag{6.4.14}$$

where

$$\begin{aligned} R_1 = {} & \alpha_2^2 \phi(-l) \frac{\partial}{\partial \xi} G_1(-1-0, -l, t) \\ & - \frac{1}{k_1 k_2} [k_1 \dot{\phi}(-1-0) - k_2 \dot{\phi}(-1+0)] G_1(-1-0, -1-0, t) \\ & + \int_{-\infty}^{-l} \alpha^2(\xi) \ddot{\phi}(\xi) G_1(-1-0, \xi, t)\, d\xi \\ & + \alpha_2^2 \int_0^t F_2(\tau) \frac{\partial}{\partial t} G_1(-1-0, y(\tau), t-\tau)\, d\tau \\ & - \int_0^t f_2(\tau) \left(\alpha_2^2 \frac{\partial}{\partial \xi} - \dot{y}(\tau) \right) \frac{\partial}{\partial t} G_1(-1-0, y(\tau), t-\tau)\, d\tau. \end{aligned} \tag{6.4.15}$$

Further, differentiating $(6.4.11_2)$ with respect to x, using the conjugacy of G_1 and G_1^* and carrying out integration by parts wherever necessary, we obtain for $x \to -1+0$

$$F_1(t) = R_2(t \mid f_2, F_2, y), \tag{6.4.16}$$

where

$$\begin{aligned} R_2 = {} & -\phi(-l) G_1^*(-1+0, -l, t) + \int_{-\infty}^{-l} \dot{\phi}(\xi) G_1^*(-1+0, \xi, t)\, d\xi \\ & - \alpha_2^2 \int_0^t F_2(\tau) \frac{\partial}{\partial \xi} G_1^*(-1+0, y(\tau), t-\tau)\, d\tau \\ & - \int_0^t f_2(\tau) \frac{d}{d\tau} G_1^*(-1-0, y(\tau), t-\tau)\, d\tau. \end{aligned} \tag{6.4.17}$$

We now turn to the coordinates (ξ, τ). We have

$$u(\xi, \tau) = U(\xi, \tau \mid f_1, F_1, F_3, z, \dot{z}, q), \tag{6.4.18}$$

---

[1] See Supplement 2, §2. Here

$$k^*(\xi) = \begin{cases} k_1 & \text{for } -1 < \xi < \infty; \\ k_2 & \text{for } -\infty < \xi < -1. \end{cases}$$

where for $z(\tau) - 1 < \xi < z(\tau),\ \tau > 0$

$$
\begin{aligned}
U = & \int_{l-1}^{0} \Phi(\xi_1)\, G_0(\xi, \xi_1, \tau)\, d\xi_1 \\
& - \alpha_2^2 \int_0^{\tau} F_1(\tau_1)\, G_0(\xi, z(\tau_1), \tau - \tau_1)\, d\tau_1 \\
& - \alpha_3^2 \int_0^{\tau} F_3(\tau_1)\, G_0(\xi, z(\tau_1), \tau - \tau_1)\, d\tau_1 \\
& + \int_0^{\tau} f_1(\tau_1) \left(\alpha_2^2 \frac{\partial}{\partial \xi} - \dot{z}(\tau_1) \right) G_0(\xi, z(\tau_1) - 1, \tau - \tau_1)\, d\tau_1 \\
& - \int_0^{\tau} \psi(\tau_1) \left(\alpha_3^2 \frac{\partial}{\partial \xi_1} - \dot{z}(\tau_1) \right) G_0(\xi, z(\tau_1), \tau - \tau_1)\, d\tau_1 \\
& + \int_0^{\tau} d\tau_1 \int_{z(\tau_1)-1}^{z(\tau_1)} q(\xi, \tau_1)\, G_0(\xi, \xi_1, \tau - \tau_1)\, d\tau_1.
\end{aligned}
\tag{6.4.19}
$$

Hence in the limit for $\xi \to -0$ we have

$$
f_2(\tau) = R_3(\tau \mid f_1, F_1, F_3, z, \dot{z}, q),
\tag{6.4.20}
$$

where

$$
\begin{aligned}
R_3 = & \int_{l-1}^{0} \Phi(\xi_1)\, G_0(-0, \xi_1, \tau)\, d\xi_1 \\
& - \alpha_2^2 \int_0^{\tau} F_1(\tau_1)\, G_0(-0, z(\tau_1) - 1, \tau - \tau_1)\, d\tau_1 \\
& + \int_0^{\tau} f_1(\tau_1) \left(\alpha_2^2 \frac{\partial}{\partial \xi_1} - \dot{z}(\tau_1) \right) G_0(-0, z(\tau_1) - 1, \tau - \tau_1)\, d\tau_1 \\
& + \alpha_3^2 \int_0^{\tau} F_3(\tau_1)\, G_0(-0, z(\tau_1), \tau - \tau_1)\, d\tau_1 \\
& - \int_0^{\tau} \psi(\tau_1) \left(\alpha_3^2 \frac{\partial}{\partial \xi_1} - \dot{z}(\tau_1) \right) G_0(-0, z(\tau_1), \tau - \tau_1)\, d\tau_1 \\
& + \int_0^{\tau} d\tau_1 \int_{z(\tau_1)-1}^{z(\tau_1)} q(\xi_1, \tau_1)\, G_0(-0, z(\tau_1), \tau - \tau_1)\, d\tau_1.
\end{aligned}
\tag{6.4.21}
$$

Furthermore, differentiating (6.4.19) with respect to ξ for $z(\tau) - 1 < \xi < z(\tau),\ \tau > 0$ and using the equations

$$
\Phi(l) = \psi(0); \quad \Phi(l-1) = f_1(0),
\tag{6.4.22}
$$

which are implied by (6.4.7) and (6.4.3), we obtain, as in our deduction

of (6.4.16),

$$q(\xi,\tau) = Q(\xi,\tau|\dot{f}_1, F_1, F_3, z, \dot{z}, q), \tag{6.4.23}$$

where for $z(\tau) - 1 < \xi < z(\tau)$

$$\begin{aligned} Q = &\int_{l-1}^{0} \dot{\Phi}(\xi_1)\, G_0^*(\xi,\xi_1,\tau)\, d\xi_1 \\ &+ a_2^2 \int_0^\tau F_1(\tau_1) \frac{\partial}{\partial \xi_1} G_0^*(\xi, z(\tau_1) - 1, \tau - \tau_1)\, d\tau_1 \\ &- \int_0^\tau \dot{f}_1(\tau_1)\, G_0^*(\xi, z(\tau_1) - 1, \tau - \tau_1)\, d\tau_1 \\ &- a_3^2 \int_0^\tau F_3(\tau_1) \frac{\partial}{\partial \xi_1} G_0^*(\xi, z(\tau_1), \tau - \tau_1)\, d\tau_1 \\ &+ \int_0^\tau \dot{\psi}(\tau_1)\, G_0^*(\xi, z(\tau_1), \tau - \tau_1)\, d\tau_1 \\ &- \int_0^\tau d\tau_1 \int_{z(\tau_1)-1}^{z(\tau)} q(\xi_1,\tau_1) \frac{\partial}{\partial \xi_1} G_0^*(\xi,\xi_1,\tau - \tau_1)\, d\xi_1. \end{aligned} \tag{6.4.24}$$

Hence in the limit for $\xi \to -0$

$$F_2(\tau) = R_4(\tau|\dot{f}_1, F_1, F_3, z, \dot{z}, q), \tag{6.4.25}$$

where

$$R_4 = Q|_{\xi=-0}. \tag{6.4.26}$$

Further, from (6.4.24) it follows, by virtue of the gap theorem for the heat potential of a double layer, that

$$F_3(\tau) = R_5(\tau|\dot{f}_1, F_1, F_3, z, \dot{z}, q). \tag{6.4.27}$$

Here

$$R_5 = 2Q|_{\xi=z(\tau)}. \tag{6.4.28}$$

Thus problem (6.4.1$_i$) ($i = 1, 2, 3, 4$) reduces to the system

$$\begin{aligned} &f_1(t) = R_1(t|f_2, F_2, y); \quad f_1(0) = \phi(-1); \quad F_1(t) = R_2(t|f_2, F_2, y); \\ &f_2(\tau) = R_3(\tau|f_1, F_1, F_3, z, \dot{z}, q); \quad F_2(\tau) = R_4(\tau|\dot{f}_1, F_1, F_3, z, \dot{z}, q); \\ &F_3(\tau) = R_5(\tau|\dot{f}_1, F_1, F_3, z, \dot{z}, q); \\ &q(\xi,\tau) = Q(\xi,\tau|\dot{f}_1, F_1, F_3, z, \dot{z}, q); \\ &\dot{y}(t) = -\gamma F_2(t); \quad y(0) = -l; \quad z(\tau) = -y(t); \quad \tau = t; \quad \xi = x - y(t), \end{aligned} \tag{6.4.29}$$

the available solution of which can be used to define $v(x,t)$ by quadrature (6.4.10). It is easily seen that for the system (6.4.29) every estimate obtained in Chapter I can be extended. Thus its solution exists, is unique, and is stable in the sense of Theorem 4. The equivalence of the construction of the solution to the solution of the original problem can be verified without difficulty as was done in Chapters I and II, and for this reason we will not enter into this problem.

CHAPTER VII

THE SINGLE-PHASE STEFAN PROBLEM FOR A GENERAL ONE-DIMENSIONAL EQUATION OF PARABOLIC TYPE. THE GENERALIZED SOLUTION OF THE STEFAN PROBLEM

§1. The single-phase Stefan problem. Kyner's existence and uniqueness theorem [86]

Every form of the existence theorem presented above was effectively based on the analysis of the algorithm for constructing the solution. We now discuss the existence theorem of Kyner, which is based on the Schauder fixed point theorem and which is ineffective from a computational point of view. Thus we examine the following problem of Kyner: find $u(x,t)$ and $y(t)$ such that u and u_x are continuous in the closure $\overline{D}$ of the parabolic domain

$$D \equiv D(0,y;0,\infty) = \{0 < x < y(t); \quad 0 < t < \infty\}$$

and satisfy the conditions

$$(pu_x)_x = fu_t; \qquad (x,t) \in D; \tag{7.1.1$_1$}$$

$$p_0 u_x = -a; \qquad x = 0; \qquad t > 0; \tag{7.1.1$_2$}$$

$$u = 0; \qquad x = y(t); \qquad t > 0; \tag{7.1.1$_3$}$$

$$y(0) = 0; \tag{7.1.1$_4$}$$

$$\dot{y}(t) = b - pu_x; \qquad x = y(t); \qquad t > 0. \tag{7.1.1$_5$}$$

Here

$$b = \text{const} > 0; \qquad a = a(t); \qquad p = p(x,t,u); \qquad p_0 \equiv p_0(t) = p(0,t,u); \qquad f = f(x,t,u). \tag{7.1.2}$$

We assume that p, f and a are defined, continuous and bounded, together with their partial derivatives of first and second order with respect to each of their arguments in the domain

$$\Omega = \{0 < x < \infty; \ 0 < t < \infty; \ -\infty < u < \infty\}, \tag{7.1.3}$$

and moreover, that throughout Ω

$$0 < \alpha_0 \leqq \frac{a(t)}{p_0(t)} \equiv b(t) < a^0; \tag{7.1.4$_1$}$$

$$0 < f_0 \leqq f(x,t,u); \tag{7.1.4$_2$}$$

$$0 < p_1 \leqq p(x,t,u); \tag{7.1.4$_3$}$$

in addition

$$p_x \geqq 0; \quad p_u \geqq 0; \quad \gamma \equiv \frac{d}{du}\frac{p_u}{f} \geqq 0. \tag{7.1.4$_4$}$$

Here α_0, a^0, f^0 and p_1 are suitable constants. It is assumed that γ can equal zero only when p and f are independent of u. In the other case γ is a negative constant different from zero.

We will prove the following theorems of Kyner.

THEOREM 18. *The solution to problem* (7.1.1$_1$)—(7.1.1$_5$) *exists on an arbitrary preassigned interval* $(0, T)$.

THEOREM 19. *The solution to problem* (7.1.1$_1$)—(7.1.1$_5$) *is unique.*

Kyner's proof of the existence theorem is based on the following arguments.

A. For a given boundary curve $x = y(t)$, continuous and reducing to zero for $t = 0$, we shall call the problem of defining $u(x,t)$, satisfying conditions (7.1.1$_1$)—(7.1.1$_3$), the subsidiary problem.[1] We have

LEMMA 15. *Let there be given an arbitrary fixed* $T > 0$ *and a function* $y(t)$ *satisfying the Lipschitz condition for* $0 < t < T$ *and reducing to zero for* $t = 0$. *Then for* $0 < t < T$ *there exists a unique solution* $u(x,t;y)$ *to the subsidiary problem.*

B. We set

$$F(x,t,u) = \int_0^u f(x,t,s)\,ds; \quad H(x,t,u) = -\int_0^u f_t(x,t,s)\,ds. \tag{7.1.5}$$

With this notation equation (7.1.1$_1$) reduces to

$$(pu_x)_x - \frac{dF}{dt} = H. \tag{7.1.6}$$

[1] In order to express the dependence of U on the boundary y we will write $U(x,t;y)$ in place of $U(x,t)$.

We now note that from $(7.1.1_3)$ and $(7.1.1_4)$ it follows that

$$\int_0^t d\tau \int_0^{y(\tau)} \frac{dF}{d\tau}\, dx = \int_0^t d\tau \cdot \frac{d}{d\tau} \int_0^{y(\tau)} F dx - \int_0^t F|_{x=y(t)} \cdot y(\tau)\, d\tau \equiv \int_0^{y(t)} F(x,t,u(x,t,y))\, dx. \tag{7.1.7$_1$}$$

Analogously from $(7.1.1_2)$, $(7.1.1_4)$ and $(7.1.1_5)$ it follows that

$$\int_0^t d\tau \int_0^{y(\tau)} (pu_x)_x dx = \int_0^t [b + a(s)]ds - y(t). \tag{7.1.7$_2$}$$

Integrating (7.1.6) and using $(7.1.7_1)$ and $(7.1.7_2)$ we obtain

$$y(t) = \int_0^t [b + a(s)]ds - \int_0^{y(t)} F(x,t,u(x,t,y))\, dx - \int_0^t d\tau \int_0^{y(\tau)} H(x,\tau;u(x,\tau;y))\, dx. \tag{7.1.8}$$

We now assert the converse:

LEMMA 16. *Let $y(t)$ be continuous, and let it reduce to zero for $t = 0$; and suppose that there exists a solution $u(x,t;y)$ to the subsidiary problem such that u and y are related by* (7.1.8). *Then u and y satisfy the relation* (7.1.15), *i.e. they constitute a solution to the original problem* $(7.1.1_1)$—(7.1.15).

We now examine the set $\mathfrak{M}$ of all functions z satisfying the conditions of Lemma 15, and the set $\mathfrak{M}^*$ of corresponding solutions of the subsidary problem. From Lemma 16 we see that to prove Theorem 18 it suffices to prove the existence in $\mathfrak{M}$ of a fixed point of the mapping

$$Rz = \int_0^t [b + a(s)]ds - \int_0^{z(t)} F[x,t,u(x,t;y)]dx - \int_0^t d\tau \int_0^{y(t)} H(x,\tau,u(x,\tau;y))\, dx. \tag{7.1.9}$$

The existence of a fixed point of the mapping is a consequence of the following lemmas.

LEMMA 17. *Let there be given an arbitrary fixed $T \geqq 0$. There exist numbers a^0 and b^0 such that for any $z \in \mathfrak{M}$ a solution $u(x,t;z)$ of the subsidiary problem can be found, satisfying for $0 \leqq x \leqq z(t)$ and $t \in (0,T)$ the inequalities*

$$0 \leqq u(x,t;z) \leqq a^0(z(t) - x); \tag{7.1.10$_1$}$$

$$-b^0 \leqq p(x,t,u)\, u_x \leqq 0. \tag{7.1.10$_2$}$$

LEMMA 18. *Let a^0 be defined as in Lemma* 17, *and let z_1 and z_2 be arbitrary elements of $\mathfrak{M}$. Set*

$$z(t) = \min\{z_1(t);\ z_2(t)\}. \tag{7.1.11}$$

Then for every $t \in (0, T)$

$$|u(z(t),t;z_1) - u(z(t),t;z_2)| \leqq a^0 |z_1(t) - z_2(t)|; \tag{7.1.12$_1$}$$

$$|u(x,t;z_1) - u(x,t;z_2)| \leqq a^0 \max_{s\in(0,t)} |z_1(s) - z_2(s)|. \tag{7.1.12$_2$}$$

In (7.1.12$_2$), $x \in (0, z(t))$.

We assume that Lemmas 17 and 18 are proved, and on this basis we now prove the existence of a fixed point of the mapping R. With this aim we choose an arbitrary fixed $T > 0$ and correspondingly, from Lemma 17, the values a^0 and b^0. Consider the subset S of $\mathfrak{M}$ consisting of all differentiable functions $z \in \mathfrak{M}$ which satisfy the inequality

$$b \leqq z \leqq b + b^0. \tag{7.1.13}$$

From (7.1.13) and the fact that $z(0) = 0$ it follows that S is uniformly bounded and equicontinuous. Consequently the closure $\overline{S}$ of S will be compact in the uniform norm

$$\|z\| = \max_{0\leq t\leq T} |z(t)| \tag{7.1.14}$$

of the space C. In addition, it is obvious that $\overline{S}$ is convex, since from $z_i \in S (i = 1, 2)$ and $0 < r < 1$ it follows that $z = rz_1 + (1 - r) z_2$ satisfies (7.1.13) and consequently belongs to S.

From the compactness and convexity of S it follows, by virtue of the Schauder fixed point theorem [**105**], that for the proof of existence of a fixed point for the mapping R it is sufficient to prove

LEMMA 19. *R defines a continuous mapping of $\overline{S}$ into itself.*

We prove Lemma 19. Let $u(x,t;z)$ be an arbitrary element of $\mathfrak{M}^*$ and let u be defined in the domain $0 \leqq x \leqq z$. We define u in the domain $z < x < \infty$, setting

$$u(x,t,z) \equiv 0; \qquad x > z; \qquad t > 0. \tag{7.1.15}$$

Now let z be an arbitrary element of $\overline{S}$ and $\{z_n\}$ a sequence of points

in S converging to z in the norm of C, i.e.

$$\lim_{n\to\infty} \|z - z_n\| = 0. \tag{7.1.16}$$

We set

$$\begin{aligned} j_n(t) &= \min\{z(t), z_n(t)\}; \\ k_n(t) &= \max\{z(t), z_n(t)\}; \end{aligned} \qquad 0 \leqq t \leqq T. \tag{7.1.17}$$

Let $u(x,t;z)$ and $u(x,t;z_n)$ be solutions to the subsidiary problem corresponding to z and z_n. Then by Lemma 18

$$|u(x,t;z) - u(x,t;z_n)| \leqq a^0 \|z - z_n\| \text{ for } 0 \leqq x \leqq j_n(t); \quad 0 \leqq t \leqq T. \tag{7.1.18}$$

At the same time, Lemma 17 and equation (7.1.15) imply

$$|u(x,t;z) - u(x,t;z_n)| \leqq a^0 |k_n(t) - x| \text{ for } j_n(t) < x < k_n(t); \quad 0 \leqq t \leqq T. \tag{7.1.19}$$

We now set

$$\begin{aligned} \Delta F_n &= |F(x,t,u(x,t;z)) - F(x,t,u(x,t;z_n))|; \\ \Delta H_n &= |H(x,t,u(x,t;z)) - H(x,t,u(x,t;z_n))|. \end{aligned} \tag{7.1.20}$$

From the definition (7.1.5) it follows that

$$F(x,t,0) = H(x,t,0) = 0. \tag{7.1.21}$$

Hence from (7.1.15) and the definition (7.1.9) of the transformation R it follows that

$$\begin{aligned} |Rz - Rz_n| \leqq &\int_0^{j_n(t)} \Delta F_n dx + \int_{j_n(t)}^{k_n(t)} \Delta F_n dx \\ &+ \int_0^t d\tau \left\{ \int_0^{j_n(\tau)} \Delta H_n dx + \int_{j_n(\tau)}^{k_n(t)} \Delta H_n dx \right\}. \end{aligned} \tag{7.1.22}$$

By assumption F_u and H_u are uniformly bounded everywhere in their domain of definition. Consequently there exists a positive constant M such that

$$\begin{aligned} \Delta F_n &\leqq M|u(x,t;z) - u(x,t;z_n)|; \\ \Delta H_n &\leqq M|u(x,t;z) - u(x,t;z_n)|. \end{aligned} \tag{7.1.23}$$

Inserting (7.1.23) into (7.1.22) and using (7.1.18) and (7.1.19), we obtain

$$\|Rz - Rz_n\| \leqq Ma^0(1+T)\{\|j_n(t)\| \cdot \|z - z_n\| + \tfrac{1}{2}\|z - z_n\|^2\}. \tag{7.1.24}$$

Thus

$$\lim_{n\to\infty} \|Rz - Rz_n\| = 0, \tag{7.1.25}$$

We now show that R maps $\overline{S}$ into itself. In fact, let $z \in S$. Then, by virtue of (7.1.9), $Rz = 0$ for $t = 0$. Obviously Rz is differentiable. Differentiating (7.1.9) and using the identity $u(z(t); t; z) \equiv 0$, we obtain

$$\begin{aligned} \frac{d}{dt} Rz = b + a(t) - \int_0^{z(t)} \frac{d}{dt} F(x,t,u(x,t;z))\,dx \\ - \int_0^{z(t)} H(x,t,u(x,t;z))\,dx. \end{aligned} \tag{7.1.26}$$

Hence, inserting (7.1.6), integrating and using $(7.1.1_2)$, we obtain

$$\frac{d}{dt} Rz = b - (pu_x)|_{x=z(t)}. \tag{7.1.27}$$

Using $(7.1.10_2)$, we find that

$$b \leqq \frac{d}{dt} Rz \leqq b + b^0. \tag{7.1.28}$$

But this shows that $Rz \in S$. Since $\overline{S}$ is closed and R is continuous on $\overline{S}$, it follows that

$$R\overline{S} \subset \overline{S}, \tag{7.1.29}$$

which was to be proved.

Thus for the proof of Theorem 18 it remains to prove Lemmas 15—18.

We first consider Lemma 16. Let $y(t) \in \overline{S}$ be a fixed point under the mapping R, and let $u(x,t)$ be the corresponding solution to the original problem. Then u and y are related by (7.1.8). Choose any h and form

$$\Delta y = y(t+h) - y(t). \tag{7.1.30}$$

By virtue of (7.1.8) we find that

$$\begin{aligned} \Delta y = \int_t^{t+h} \left[b + a(s) - \int_0^{y(s)} H(x,s,u(x,s))\,dx \right] ds \\ - \int_0^{y(t)} dx \int_t^{t+h} \frac{d}{ds} F(x,s,u(x,s))\,ds \\ - \int_{y(t)}^{y(t+h)} F(x,t+h,u(x,t+h))\,dx. \end{aligned} \tag{7.1.31}$$

Using (7.1.6) and (7.1.1_2), we obtain

$$\int_0^{y(t)} dx \int_t^{t+h} \frac{d}{ds} F(x,s,u(x,s))\, ds$$

$$= \int_t^{t+h} p(y(t),s,u)\, u_x(y(t),s)\, ds \tag{7.1.32}$$

$$+ \int_t^{t+h} a(s)\, ds - \int_t^{t+h} ds \int_0^{y(t)} H(x,s,u(x,s))\, dx.$$

Substituting this into (7.1.31), we see that

$$\Delta y + \int_{y(t)}^{y(t+h)} F(x,t,u(x,t))\, dx \tag{7.1.33}$$

$$= \int_t^{t+h} (b - pu_x)|_{x=y(t)}\, ds + \int_t^{t+h} ds \int_{y(s)}^{y(t)} H(x,s,u(x,s))\, ds$$

and by virtue of the mean value theorem

$$\Delta y \cdot [1 + F(\xi,t,u(\xi,t))] \tag{7.1.34}$$

$$= \int_t^{t+h} \left\{ (b - pu_x)|_{x=y(t)} + \int_{y(s)} H(x,s,u(x,s))\, dx \right\} ds.$$

Here $\xi \in (y(t), y(t+h))$.

The integrand function on the right side of (7.1.34) is continuous. In addition,

$$\lim_{\xi \to y(t)} F(\xi,t,u(\xi,t)) = F(y(t),t,0) = 0. \tag{7.1.35}$$

Consequently, the limit

$$\lim_{h \to 0} \frac{\Delta y}{h} = b - (pu_x)|_{x=y(t)} \tag{7.1.36}$$

exists, which completes our proof.

We now prove Lemma 17. Let u and z satisfy its conditions. Obviously $u \geqq 0$. In fact, if not then by virtue of the maximum principle u attains its negative minimum for $x = 0$; but this is impossible since $u_x|_{x=0} < 0$ by (7.1.1_2) and (7.1.4_i) $(i = 1,2,3,4)$.

Thus $u > 0$ in D. But by (7.1.1_3) and the continuity of u_x in $\overline{D}$ this shows that

$$u_x|_{x=y(t)} \leqq 0. \tag{7.1.37}$$

Hence $u_x \leqq 0$ everywhere in D. We introduce the notation

$$u_x = v; \qquad \frac{p(x,t,u)}{f(x,t,u)} = q(x,t) > 0;$$

$$\frac{1}{f(x,t,u)} \frac{d}{dx} p(x,t,u) = q^*(x,t). \tag{7.1.38}$$

Then $(7.1.1_1)$ assumes the form

$$qv_x + q^* v = u_t. \tag{7.1.39}$$

Differentiating with respect to x, we obtain

$$qv_{xx} + \left(q^* + \frac{d}{dx} q\right) v_x + v \frac{d}{dx} q^* = v_t. \tag{7.1.40}$$

Thus v satisfies a homogeneous equation of parabolic type. We note that dq^*/dx is bounded, since $f \geqq f_0 > 0$ by condition $(7.1.4_2)$. We set

$$w = \exp(-\lambda t)\, v. \tag{7.1.41}$$

From (7.1.40) it follows that

$$qw_{xx} + \left(q^* + \frac{d}{dx} q\right) w_x - \left(\lambda - \frac{d}{dx} q^*\right) w = w_t. \tag{7.1.42}$$

By the boundedness of dq^*/dx there exists some λ so large that

$$\lambda - \frac{d}{dx} q^* > 0. \tag{7.1.43}$$

For a chosen λ this permits us to apply the maximum principle according to which the positive maximum of w is attained on the boundary of the domain D. But by (7.1.37) and $(7.1.1_2)$

$$w \leqq 0 \quad \text{for} \quad x = 0 \quad \text{and for} \quad x = y(t); \qquad t > 0. \tag{7.1.44}$$

Thus everywhere in D we have $w \leqq 0$, and together with this

$$u_x(x,t) \leqq 0; \qquad (x,t) \in D, \tag{7.1.45}$$

which completes the proof.

We now estimate $u_x(x,t)$ from below. We first show that

$$u_x(0,t) \geqq -a^0. \tag{7.1.46}$$

We now determine $U(x,t)$ by the conditions: U and U_x are continuous in the closed domain $D(0,y;0,T)$, and moreover

$$(7.1.47_1) \qquad (pU_x)_x = fU_t; \qquad (x,t) \in D;$$

$$(7.1.47_2) \qquad U = 0 \quad \text{for} \quad x = y(t); \quad t > 0;$$

$$(7.1.47_3) \qquad U_x = -a^0 \quad \text{for} \quad x = 0; \quad t > 0.$$

By virtue of Lemma 15, U exists and is unique.

Repeating the proof of (7.1.45), we see that

$$(7.1.48) \qquad U - u \geqq 0 \quad \text{for} \quad (x,t) \in \overline{D}.$$

Hence, since $U = u = 0$ for $x = y(t)$ and $t > 0$, it follows that

$$(7.1.49) \qquad U_x(y(t),t) \leqq u_x(y(t),t).$$

Furthermore, set

$$(7.1.50) \qquad V = U + a^0 x.$$

Substituting this into $(7.1.47_1)$, we find that

$$(7.1.51) \quad pV_{xx} + (p_x + p_u V_x - 2a^0 p_u)\, V_x - fV_t = a^0(p_x - a^0 p_u).$$

By virtue of conditions $(7.1.4_i)$ $(i = 1,2,3,4)$

$$(7.1.52) \qquad p > 0; \qquad f > 0; \qquad p_x - a^0 p_u > 0.$$

Consequently we can apply the maximum principle to V; hence V attains its positive maximum on the boundary of D. But by $(7.1.47_3)$

$$(7.1.53) \qquad V_x = 0 \quad \text{for} \quad x = 0; \qquad t > 0.$$

Hence from the theorem of Friedman it follows that the positive maximum of V is not attained for $x = 0$, and so it must be attained for $x = y(t)$. But

$$(7.1.54) \qquad V|_{x=y(t)} = a^0 y(t).$$

Consequently from the monotonicity of $y(t)$ it follows that

$$(7.1.55) \qquad V(x,t) \leqq a^0 y(t); \qquad (x,t) \in D.$$

Comparing (7.1.55), (7.1.50) and (7.1.48) proves that

$$(7.1.56) \qquad 0 \leqq u(x,t) \leqq a^0(y(t) - x) \quad \text{for} \quad 0 \leqq x \leqq y(t); \quad 0 \leqq t \leqq T,$$

which establishes the first assertion of Lemma 17.

We now prove the second assertion of the lemma. With this aim we use the fact that the maximum of v in $\overline{D}$ is attained on $x = y(t)$, to show that (7.1.54) implies

$$(7.1.57) \qquad V_x(y(t),t) \geqq 0,$$

or equivalently

(7.1.58) $$U_x(y(t),t) \geqq -a^0.$$

Inserting (7.1.58) into (7.1.49) and using (7.1.46) and (7.1.45), we obtain the estimates

(7.1.59) $$-a^0 \leqq u_x(x,t) \leqq 0 \quad \text{for} \quad x = y(t); \quad x = 0; \quad 0 \leqq t \leqq T.$$

Passing from u_x to w by the use of equations (7.1.38) and (7.1.41), we find by the extensibility of the maximum principle to w that

(7.1.60) $$-a^0 \leqq w \leqq 0; \qquad (x,t) \in D.$$

Hence for $0 \leqq x \leqq y(t)$, $0 \leqq t \leqq T$

(7.1.61) $$-a^0 \exp(\psi T) \leqq u_x \leqq 0.$$

By the boundedness of p in Ω we have

(7.1.62) $$|p| < N; \qquad (x,t,u) \in \Omega.$$

Setting

(7.1.63) $$b^0 = a^0 \exp(\lambda T)\, N,$$

we find that

(7.1.64) $$-b^0 \leqq p(x,t,u)\, u_x \leqq 0; \quad 0 \leqq x \leqq y(t); \quad 0 \leqq t \leqq T,$$

which coincides with (7.1.10$_2$). In order to prove that this coincidence is not formal, and at the same time to complete the proof of Lemma 17, it is sufficient to prove the independence of b^0, or equivalently the independence of λ, from z. With this aim we show that the equation (7.1.40) can be written in the form

(7.1.65) $$Av_{xx} + Bv_x + Cv = v_t,$$

where

$$A = \frac{p}{f}; \quad B = \frac{d}{dx}\left(\frac{p}{f}\right) + \frac{p_x + p_u v + p_u}{f};$$

$$C = \alpha + \beta v + \gamma v^2;$$

(7.1.66)

$$\alpha = \left(p_{xx} - p_x \frac{f_x}{f}\right) \cdot \frac{1}{f}; \quad \beta = \frac{2p_{ux}}{f} - \frac{p_u f_x + pf}{f^2};$$

$$\gamma = \frac{d}{du}\left(\frac{p_u}{f}\right).$$

By the proof $v < 0$. Moreover, since $p > 0$ we have $f > 0$. Consequently $A > 0$. Furthermore, if p or f is dependent on u, then by condition $(7.1.4_4)$ $\gamma \leqq -\delta < 0$. In this case C can be examined by a third quadrature of the coefficients α, β and γ, uniformly bounded for $0 < x, t < \infty$. Since γ is different from zero, C is bounded above by a constant λ independent of z. But this shows that in (7.1.63) λ is not dependent on z, which completes the proof. If p and f are both dependent on u, then $\beta = \gamma \equiv 0$ and consequently C is not dependent on v. But this again signifies the independence of λ from z. Thus the lemma is proved.

We now prove Lemma 18. Let $z_i(t) \in \mathfrak{M}$ $(i = 1, 2)$ and let $u(x, t; z_i)$ be a solution to the subsidiary problem corresponding to z_i. We set

$$\begin{aligned} w(x,t) &= u(x,t;z_1) - u(x,t;z_2); \\ z(t) &= \min\{z_1(t);\ z_2(t)\}. \end{aligned} \tag{7.1.67}$$

By virtue of Lemma 17

$$0 \leqq u(x,t;z_i) \leqq a^0(z_i(t) - x) \quad \text{for} \quad 0 \leqq x \leqq z_i(t); \quad 0 \leqq t \leqq T \quad (i = 1,2). \tag{7.1.68$_1$}$$

In addition

$$u(z_i(t), t; z_i) \equiv 0 \qquad (i = 1, 2). \tag{7.1.68$_2$}$$

Suppose for definiteness that $z(t) = z_2(t)$. Then

$$\begin{aligned} 0 \leqq w(z(t),t) &= u(z_2(t),t;z_1(t)) - u(z_2(t),t;z_2(t)) \\ &\equiv u(z_2(t),t;z_1(t)) \leqq a^0(z_1(t) - z_2(t)), \end{aligned} \tag{7.1.69}$$

which proves (7.1.12).

Further, from $(7.1.1_1)$ it follows that $w(x, t)$ satisfies a homogeneous equation of parabolic type, to which we can apply the maximum principle. We note that, by virtue of the theorem of Friedman, neither a maximum nor a minimum of w can be attained for $x = 0$. Since here $w_x = 0$, we find that the maximum modulus of w is assumed for $x = z(t)$; hence by virtue of (7.1.69) this is equivalent to $(7.1.12_2)$. The lemma is proved.

We now prove Lemma 15. We first note that if the domain $D_t = \{0 < x < y(t)\}$ does not degenerate to a point for $t = 0$, then the validity of the lemma follows, as above, from the general theory of the boundary problem for a quasilinear equation of parabolic type, i.e. from the classical result of Gevrey [**50**]. However, if D_t degenerates to a point

for $t = 0$ we cannot apply the above work. Thus we do the following. We set

$$v(x,t) = u(x,t) + b(t)\,x; \tag{7.1.70}$$

$$q(x,t,v) \equiv p(x,t,u); \qquad \phi(x,t,v) \equiv f(x,t,u). \tag{7.1.71}$$

Then v is a solution to the boundary problem

$$\begin{aligned} &\frac{\partial}{\partial x}\left[q\left(\frac{\partial}{\partial x} - b(t)\right)\right] v = \phi \frac{\partial v}{\partial t}; \quad (x,t) \in D; \\ &\left.\frac{\partial v}{\partial x}\right|_{x=1} = 0; \qquad v|_{x=z} = b(t)\,z(t). \end{aligned} \tag{7.1.72}$$

Now we extend q and ϕ evenly with respect to x to the domain $D^* = \{-z(\tau) < x < z(\tau);\ 0 < \tau \leqq t \leqq T\}$, and examine the solution w to the boundary problem

$$\frac{\partial}{\partial x}\left[q\left(\frac{\partial}{\partial x} - b(t)\right)\right] w = \phi \frac{\partial w}{\partial t}; \qquad (x,t) \in D^*; \tag{7.1.73}$$

$$w|_{|x|=z} = b(t)\,z(t). \tag{7.1.74}$$

It is obvious that if w exists, then it is even with respect to x.

We choose an arbitrary sequence

$$T > t_0 > t_1 > \cdots > t_n > \cdots; \qquad \lim_{n\to\infty} t_n = 0 \tag{7.1.75}$$

and define $w_n(x,t)$ as a solution to the boundary problem

$$\frac{\partial}{\partial x}\left[q\left(\frac{\partial}{\partial x} - b(t)\right)\right] w_n = \phi \frac{\partial w_n}{\partial t}; \quad |x| < z(t); \quad t_n < t \leqq T; \tag{7.1.76}$$

$$w_n|_{|x|=z} = b(t)\,z(t); \qquad w_n|_{t=t_n} = \psi_n(x), \tag{7.1.77}$$

where $\{\psi_n\}$ is an arbitrary sequence of even functions, defined and twice continuously differentiable in the domain $-z(t_n) < x < z(t_n)$, reducing to $b(t_n)\,z(t_n)$ for $x = z(t_n)$ and such that the sequence $\{\dot{\psi}_n\}$ is uniformly bounded. As we noted above, the existence of w_n for arbitrary n follows from the classical result of Gevrey. Obviously $\{w_n\}$ is uniformly convergent. This follows from the uniform convergence of $\{\psi_n\}$ to zero and from the application of the maximum principle to $w_{m,n} = w_n - w_m$ for any m and n, as a solution of a homogeneous parabolic equation with bounded coefficients. Thus the limit $w = \lim_{n\to\infty} w_n$

exists, and in addition the convergence is uniform. It is obvious that $w = b(t)\, z(t)$ for $|x| = z(t)$ and $t > 0$. Furthermore, it is easily seen that w satisfies equation (7.1.73) within D^*. In fact, let D' be a parabolic domain situated within D^* together with its parabolic boundary Γ'. Define w' in D' as a solution to equation (7.7.73), attaining the same values on Γ' as does w. Then the sequence $\{w_n\}$ converges to w' on Γ'. Hence from the application of the maximum principle to $w_n - w'$ it follows that $\lim_{n\to\infty} w_n = w'$ throughout D'. This means that $w' \equiv w$. Thus the solution w of problem (7.1.73), (7.1.74) does in fact exist. But since w is even, this proves the existence of a solution to problem (7.7.72) and completes the proof of Lemma 15, because of the uniqueness of the solution to the subsidiary problem, which by virtue of the maximum principle does not require a special proof.[2)]

We now prove Theorem 19. Let us assume the existence in the set S of two fixed points g and h belonging to R. Repeating the proof of Lemma 19 proves the following inequality, analogous to (7.1.24):

$$E(t) \leqq (1 + T)\, Ma^0 \left[j(t) + \frac{1}{2} E(t) \right] E(t). \tag{7.1.78}$$

Here

$$E(t) = \max_{s \in [0,t]} |h(s) - g(s)|; \qquad j(t) = \min\{g(t);\ h(t)\}. \tag{7.1.79}$$

We have

$$\lim_{t\to 0} j(t) = \lim_{t\to 0} E(t) = 0. \tag{7.1.80}$$

Hence by (7.1.78) there exists some $t^0 > 0$ such that

$$g(t) \equiv h(t) \quad \text{for} \quad 0 \leqq t \leqq t^0. \tag{7.1.81}$$

Assume now that that the set $\mathscr{S}$ of all t for which $E(t) \neq 0$ is not empty. We assume that

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[2)] In his original work Kyner assumed that $p = p(x, t, u, u_x)$ and in addition $p_{u_x} \leqq 0$. The condition $\gamma = (d/du)(p_u/f) < 0$ was not formulated. The assertion that the constant b^0 of Lemma 17 is independent of z is unproved in the work of Kyner, since the independence of the constant λ in (7.1.6) of z was not proved. Similarly Lemma 15 was not entirely rigorously proved. Namely, without any further remarks Kyner proved the basis of the assumption of differentiability of $z(t)$. However, the nondifferentiability of the boundary may enter into the boundary set S. Finally, in the case of independence of p not only of u but also of u_x, referring to the results of Gevrey, the continuity of u_x everywhere up to the boundary of its domain of definition is not guaranteed.

$$T^0 = \inf \mathscr{E};$$

$$m = h(T^0) = g(T^0); \tag{7.1.82}$$

$$A(r) = \max_t |h(t) - g(t)| \quad \text{for} \quad T^0 \leqq t \leqq T^0 + r.$$

From the definition of T^0 it follows that for arbitrary $r > 0$

$$A(r) > 0. \tag{7.1.83}$$

Setting

$$\Delta u(x,t) = u(x,t;h) - u(x,t;g) \tag{7.1.84}$$

and repeating the derivations of (7.1.24) and (7.1.78), we find that for $T^0 \leqq t \leqq T^0 + r \leqq T$

$$0 < A(r) \leqq M(1+T)\left[\max_t \left(\int_0^m + \int_m^{j(t)} |\Delta u(x,t)|\,dx\right) + \frac{1}{2}a^0 A^2(r)\right]. \tag{7.1.85}$$

Using Lemma 18, we find that

$$\max_t \int_m^{j(t)} |\Delta u(x,t)|\,dx \leqq a^0(j(T^0+r) - m)\,A(r). \tag{7.1.86}$$

By assumption $A(r) > 0$. Therefore (7.1.85) can be divided by $A(r)$. Using (7.1.86), we obtain

$$1 \leqq M(1+T)\left[\max_t \int_0^m \frac{|\Delta u(x,t)|}{A(r)}\,dx + a^0\left(j(T^0+r) + \frac{1}{2}A(r)\right)\right].$$

We now show

$$\lim_{r\to\infty} \max_t \int_0^m \left|\frac{\Delta u(x,t)}{a^0 A(r)}\right| dx = 0; \qquad T^0 \leqq t \leqq T^0 + r. \tag{7.1.88}$$

With this aim we set

$$v(x,t) = \frac{\Delta u(x,t)}{a^0 A(r)}. \tag{7.1.89}$$

From (7.1.1$_2$) and (7.1.82) it follows that

$$v_x(0,t) = 0; \quad T^0 \leqq t \leqq T^0 + r; \quad v(x,T^0) = 0; \quad 0 \leqq x \leqq m. \tag{7.1.90}$$

Further, using (7.1.82) and Lemma 18, we find that

$$|v(j(t),t)| \leqq 1; \qquad T^0 \leqq t \leqq T^0 + r. \tag{7.1.91}$$

Finally, from $(7.1.1_1)$ it follows that $v(x,t)$ satisfies a homogeneous equation of parabolic type of the form

$$Av_{xx} + Bv_x + Cv = Dv_t, \tag{7.1.92}$$

where $A \geqq A_0 > 0$, $D \geqq D_0 > 0$ and C is bounded. Hence from the maximum principle it follows that for sufficiently large λ the function

$$w(x,t) = \exp(-\lambda t)\, v(x,t) \tag{7.1.93}$$

assumes its negative minimum and positive maximum on the boundary of its domain of definition. From (7.1.90) it follows that the attainment is for $x = j(t)$. Let us define

$$\max w = w(s, j(s)); \qquad \min w = w(s', j(s')). \tag{7.1.94}$$

Then from (7.1.91) it follows that

$$-\exp(-\lambda(s' - t)) \leqq v(x,t) \leqq \exp(-\lambda(s - t)). \tag{7.1.95}$$

Consequently for $r_0 = \min(s, s')$, $0 \leqq x \leqq j(t)$ and $T^0 \leqq t \leqq T^0 + r_0$ we have

$$|v(x,t)| \leqq 1. \tag{7.1.96}$$

We note that the coefficient in (7.1.92) is dependent on g and h. This equation can be considered as linear, since g and h are given. Now let w be a bounded solution to equation (7.1.92) in the domain

$$D_m = \{0 < x \leqq m; \quad 0 < t \leqq T^0 + r_0 \leqq T\} \tag{7.1.97}$$

satisfying the conditions

$$w_x(0,t) = 0; \qquad w(x,0) = 0; \qquad w(m,t) = 1. \tag{7.1.98}$$

The functions

$$z = (w \pm v) \exp(-\lambda t)$$

are not negative on the boundary of D_m, and consequently, by virtue of the maximum principle, they are nonnegative throughout D_m. Thus

$$|v(x,t)| \leqq w(x,t); \quad 0 < x < m; \quad T^0 < t \leqq T^0 + r_0. \tag{7.1.99}$$

Integrating, we find that

$$\max_t \int_0^m |v(x,t)|\,dx \leqq \max_t \int_0^m w(x,t)\,dx; \quad T^0 \leqq t \leqq T^0 + r_0. \tag{7.1.100}$$

From the boundedness of w in $\overline{D}_m$ and the continuity of w for $\delta \leqq x \leqq m - \delta$ and $T^0 \leqq t \leqq T^0 + r_0$, where $\delta > 0$ is arbitrarily small, it follows that

$$\lim_{r\to 0} \int_0^m w(x,t)\,dx = \int_0^m \lim_{r\to 0} w(x,t)\,dx = 0. \tag{7.1.101}$$

However, by virtue of (7.1.99) and (7.1.89), this shows that (7.1.88) is in fact valid.

From the continuity of g and h and equation (7.1.82) it follows that

$$\lim_{r\to 0} j(t) = \lim_{r\to 0} A(r) = 0; \qquad T^0 \leqq t \leqq T^0 + r. \tag{7.1.102}$$

Comparing (7.1.88) and (7.1.102) with (7.1.87), we obtain a contradiction.

By the same token the proof of the emptiness of the set $\mathscr{S}$, and together with this the uniqueness of the solution of the original problem on an arbitrary preassigned time interval, is completed.

§2. The generalized solution of the Stefan problem

We now present the theory of the generalized solution to the Stefan problem, developed by S. L. Kamenomostskaja [73] and extended by O. A. Oleĭnik [107]. We limit our examination to the homogeneous case in its simplest formulation. In the final paragraphs we state the most general results of [73] and [107]. For the proof of Theorem 20 (the uniqueness theorem) we use the presentation of Kamenomostskaja. For the proof of Theorem 21 (the existence theorem) we will follow the presentation of Oleĭnik with simplifications.

Suppose we are given the numbers $b_i \geqq 0$ $(i = 1, \cdots, N)$ and a function $c(u)$ which is bounded and piecewise continuous for $-\infty < u < \infty$, with jump points $u = s_i$ $(i = 1, \cdots, N)$, such that

$$c(u) \geqq c_0 = \text{const} > 0. \tag{7.2.1$_1$}$$

We denote by l_i the intervals

$$l_i = \{ s_{i-1} < u < s_i \}; \quad s_0 = -\infty; \quad s_{N+1} = +\infty \tag{7.2.1$_2$}$$

and pose the problem of finding a function $u(x,t)$, continuous in the closure $\overline{D}$ of the domain

$$D = \{ 0 < x < L;\ 0 < t < T \} \tag{7.2.1$_3$}$$

and satisfying the following conditions:

a) The function u satisfies the equation

$$(7.2.2_1) \qquad u_{xx} = c(u)u_t; \qquad (x,t) \in D; \qquad u \neq s_i;$$

throughout D, except on the curves $x = x_i(t)$ at which $u = s_i$ $(i = 1, \cdots, N)$.

b) On the boundary of D, u assumes given values

$$(7.2.2_2) \qquad u|_{t=0} = u_0(x); \qquad u|_{x=0} u_1(t); \qquad u|_{x=L} = u_2(t).$$

c) On the curves $x = x_i(t)$, which are considered smooth, we have the condition

$$(7.2.2_3) \qquad b_i \dot{x}_i(t) = u_x|_{u=s_i-0} - u_x|_{u=s_i+0}.$$

A function $u(x,t)$ satisfying these conditions will be called the classical solution of the Stefan problem.

We now select $u^0 = \text{const} < s_1$ and set

$$(7.2.3) \qquad \gamma_i = \sum_{j=1}^{i=1} b_j; \qquad i \geqq 1; \qquad \gamma_0 = 0.$$

We define the piecewise continuous function $a(u)$ by the equations

$$(7.2.4) \qquad \begin{aligned} a(u) &= \int_{u_0}^{u} c(s)\,ds + \gamma_i \quad \text{for} \quad u \in l_i; \\ a(s_i) &= \alpha_i, \end{aligned}$$

where α_i is an arbitrary point in the interval $(a(s_i - 0),\ a(s_i + 0))$. Such a definition permits us, for $u = u(x,t)$, to consider $a(u)$ as a function of the point (x,t).

By our assumptions $b_i \geqq 0$ and $c(u) \geqq c_0 > 0$ for $-\infty < u < \infty$. Consequently a is monotonically increasing with u, and moreover, for $u \neq v$ we have

$$(7.2.5_1) \qquad \frac{a(u) - a(v)}{u - v} \geqq c_0 > 0.$$

We note that

$$(7.2.5_2) \qquad a(s_i + 0) - a(s_i - 0) = b_i \qquad (i = 1, 2, 3, \cdots, N).$$

It is obvious that equation $(7.2.2_1)$ can be written in the form

$$(7.2.6) \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial a(u)}{\partial t}; \qquad (x,t) \in D; \qquad u \neq s_i.$$

Let $u(x,t)$ be the classical solution to the Stefan problem. Then it is a solution to the problem (7.2.6), (7.2.2$_2$). Choose an arbitrary function $F(x,t)$, twice continuously differentiable with respect to x and once with respect to t, which is defined in $\overline{D}$ and satisfies the conditions

$$F(0,t) = F(L,t) = F(x,T) = 0. \tag{7.2.7}$$

Multiplying (7.2.6) by F, integrating on D and using conditions (7.2.2$_2$), (7.2.2$_3$) and (7.2.5$_2$), we easily find that $u(x,t)$ satisfies the integral identity

$$\begin{aligned}\int_D [u(x,t)\,F_{xx}(x,t) + a(u(x,t))\,F_t(x,t)]\,dxdt \\ + \int_0^L a_0(x)\,F(x,0)\,dx \\ + \int_0^T [u_1(t)\,F_x(0,t) - u_2(t)\,F_x(L,t)]\,dt = 0.\end{aligned} \tag{7.2.8}$$

Here

$$a_0(x) = a[u_0(x)]. \tag{7.2.9}$$

We now state the following definition: A function $u(x,t)$ which is measurable in D is called a generalized solution to the Stefan problem (7.2.2$_i$) ($i = 1,2,3$) if the integral identity (7.2.8) is valid for an arbitrary function $F(x,t)$ defined in $\overline{D}$ which is twice continuously differentiable with respect to x and once with respect to t, and satisfies condition (7.2.7).

From the derivation of (7.2.8) it follows that the classical solution to the Stefan problem is at the same time a generalized solution to it. Therefore the introduction of the concept of generalized solution will be justified if the following theorem is valid.

THEOREM 20. *If a generalized solution to the Stefan problem exists, then it is unique.*

PROOF OF THEOREM 20. Suppose that u and v are two generalized solutions of the Stefan problem. Subtracting the integral identity for v from the integral identity for u, we obtain

$$\int_D [a(u) - a(v)]\left\{F_t + \frac{u-v}{a(u)-a(v)}F_{xx}\right\}dxdt = 0. \tag{7.2.10}$$

If for an arbitrary infinitely differentiable function $\phi(x)$ vanishing in a neighborhood of the points $x = 0$ and $x = L$ there exists a solution to the equation

$$F_t + \frac{u - v}{a(u) - a(v)} F_{xx} = \Phi, \tag{7.2.11}$$

continuous in $\overline{D}$ and satisfying the condition (7.2.7), then from (7.2.10) it follows that

$$\int_D [a(u) - a(v)]\Phi dxdt = 0. \tag{7.2.12}$$

Since the set of such functions ϕ is everywhere dense in $L_2(D)$,[3] it follows from (7.2.12) that almost everywhere in D

$$a(u) = a(v). \tag{7.2.13}$$

Hence from the monotonicity of $a(u)$ it follows that

$$u = v, \tag{7.2.14}$$

almost everywhere, which signifies the uniqueness of the generalized solution of the problem.

The classical theorem of existence for a quasilinear equation of parabolic type is valid for sufficiently smooth coefficients. Since

$$\mu(x,t) = \frac{u - v}{a(u) - a(v)} \tag{7.2.15}$$

is only a bounded measurable function in D, referring to the existence of a solution for the boundary problem (7.2.11), (7.2.7) is seemingly impossible. In this regard we note the following. From $(7.2.5_2)$ it follows that

$$\mu(x,t) \in L_2(D); \qquad 0 < \mu(x,t) \leqq \frac{1}{c_0}. \tag{7.2.16}$$

As is known [79], for any $n \geqq 1$ it is possible to find a function $\bar{\mu}_n(x,t)$ continuous in $\overline{D}$ together with its derivatives of any order, such that

$$0 < \bar{\mu}_n \leqq \frac{1}{c_0}; \qquad \|\mu - \bar{\mu}_n\| < \frac{1}{n}. \tag{7.2.17}$$

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[3] $L_2(D)$ is the space of functions which are square integrable in D. Below, we will denote the norm of $L_2(D)$ simply by $\| \|$.

We assume that

$$\mu_n = \bar{\mu}_n + \frac{1}{n}. \tag{7.2.18}$$

In this case

$$\frac{1}{n} \leqq \mu_n \leqq \frac{1}{c_0} + 1;$$

$$\|\mu - \mu_n\| \leqq \|\mu - \bar{\mu}_n\| + \|\bar{\mu}_n - \mu_n\| \leqq \frac{1}{n}(1 + (LT)^{1/2}). \tag{7.2.19$_1$}$$

Hence

$$\left\|\frac{\mu}{\mu_n}\right\| = \left\|\frac{\mu - \mu_n}{\mu_n} + 1\right\| \leqq 1 + 2(LT)^{1/2} = K. \tag{7.2.19$_2$}$$

We now define $f_n(x,t)$ as the solution to the boundary problem

$$f_{nt} + \mu_n f_{nxx} = \Phi; \qquad (x,t) \in D;$$
$$f_n|_{t=T} = f_n|_{x=0} = f_n|_{x=L} = 0. \tag{7.2.20}$$

The existence of f_n follows from the classical result of Gevrey [**50**].

We have

$$f_{nt} + \mu f_{nxx} = \Phi - (\mu_n - \mu) f_{nxx}. \tag{7.2.21}$$

Here f_n belongs to the class of functions for which the identity (7.2.10) is realized. Therefore from (7.2.21), (7.2.15) and (7.2.10) it follows that

$$\int_D [a(u) - a(v)]\Phi dxdt = \int_D (a(u) - a(v))\ (\mu_n - \mu) f_{nxx} dxdt \equiv I_0. \tag{7.2.22}$$

The left side of (7.2.22) is independent of n. Consequently for the proof of (7.2.12), and equivalently for the proof of the uniqueness theorem, it is sufficient to show that

$$\lim_{n\to\infty} I_0 = 0. \tag{7.2.23}$$

But $a(u) - a(v)$ is bounded in D. Thus (7.2.23) follows from

$$\lim_{n\to\infty} I_1 = 0. \tag{7.2.24$_1$}$$

Here

$$I_1 = \int_D |\mu - \mu_n|\ |f_{nxx}|\, dxdt. \tag{7.2.24$_2$}$$

In order to prove (7.2.24$_1$) we note that from $\mu_n > 0$ and $\mu > 0$ it follows that

$$I_1 = I_2 + I_3, \tag{7.2.25$_0$}$$

where

$$I_2 = \int_D \sqrt{\mu_n}|\sqrt{\mu}| \cdot |f_{nxx}|\, dxdt;$$
$$I_3 = \int_D \sqrt{\mu}|\sqrt{\mu_n} - \sqrt{\mu}| \cdot |f_{nxx}|\, dxdt. \tag{7.2.25$_1$}$$

By virtue of the Bunjakovskiĭ-Schwarz inequality

$$I_2^2 < \|\sqrt{\mu} - \sqrt{\mu_n}\|^2 \int_D \mu_n f_{nxx}^2\, dxdt. \tag{7.2.26}$$

We now transform the expression $\int_D \mu_n f_{nxx}^2\, dxdt$. With this aim we multiply the first of the equations (7.2.20) by f_{nxx} and integrate over D. We find that

$$\int_D \mu_n f_{nxx}^2\, dxdt = \int_D f_{nxx}\Phi dxdt - \int_D f_{nt} f_{nxx} dxdt. \tag{7.2.27}$$

It is easily seen that (7.2.20) implies

$$\int_D f_{nt} f_{nxx} dxdt = \frac{1}{2}\int_0^L f_{nx}^2 \,|_{t=0} dx. \tag{7.2.28$_1$}$$

Furthermore, from the equation $\Phi = 0$ in a neighborhood of the points $x = 0$ and $x = L$ it follows that

$$\int_D f_{nxx}\Phi dxdt = \int_D f_n \Phi_{xx} dxdt. \tag{7.2.28$_2$}$$

Inserting (7.2.28$_i$) ($i = 1, 2$) into (7.2.27), we obtain

$$\int_D \mu_n f_{nxx}^2\, dxdt = \int_D f_n \Phi_{xx} dxdt - \frac{1}{2}\int_0^L f_{nx}^2\,(x, 0)\, dx, \tag{7.2.29$_1$}$$

from which we have the estimate

$$0 \leqq \int_D \mu_n f_{nxx}^2\, dxdt \leqq \int_D f_n \Phi_{xx} dxdt. \tag{7.2.29$_2$}$$

But from the maximum principle it follows that $\{f_n\}$ is uniformly bounded with respect to n. In fact, let us choose any $\epsilon > 0$ and set

$$\phi_n = f_n \exp(\epsilon t). \tag{7.2.30}$$

We find that

(7.2.31) $$\phi_{nt} + \mu_n \phi_{nxx} - \epsilon \phi_n = \Phi \exp(\epsilon t).$$

If ϕ_n assumes a positive maximum at a point (x_0, t_0) for $0 \leqq t_0 \leqq T$, then at this point $\phi_{nt} \leqq 0$ and $\phi_{nxx} \leqq 0$. Consequently

(7.2.32) $$f_n < \frac{1}{\epsilon} \exp(\epsilon T) \max |\Phi(x,t)|.$$

We can analogously estimate f_n from below. Thus $\{f_n\}$ is uniformly bounded. Hence there exists some $K_2 > 0$ such that

(7.2.33) $$0 \leqq \int_D f_n \Phi_{xx} dxdt \leqq K_2^2.$$

Comparing (7.2.33), (7.2.29$_2$) and (7.2.26) and noting that by (7.2.19$_1$) and (7.2.18) the sequence $\{\sqrt{\mu_n}\}$ converges to $\sqrt{\mu}$ in the mean, we find that

(7.2.34) $$\lim_{n \to \infty} I_2 = 0.$$

We now investigate I_3. To do this we choose $\epsilon > 0$ and $\delta > 0$ arbitrarily small, and examine the set E_n of points $(x,t) \in \overline{D}$ such that

(7.2.35) $$|\sqrt{\mu(x,t)} - \sqrt{\mu_n(x,t)}| > \epsilon \quad \text{for} \quad (x,t) \in E_n.$$

From the mean convergence of $\{\sqrt{\mu_n}\}$ to $\sqrt{\mu}$ we find that $\{\sqrt{\mu_n}\}$ converges to $\sqrt{\mu}$ in measure [79]. Consequently it is possible to find some $n_0 > 0$ such that for every $n > n_0$

(7.2.36) $$\text{mes}\, E_n < \delta.$$

Choose a fixed $n > n_0$. We then have

(7.2.37) $$I_3 = I_3' + I_3'' = \int_{E_n} + \int_{D \setminus E_n} \sqrt{\mu}\, |\sqrt{\mu} - \sqrt{\mu_n}| \cdot |f_{nxx}|\, dxdt.$$

By virtue of the Bunjakovskiĭ-Schwarz inequality we obtain

(7.2.38) $$(I_3'')^2 \leqq \int_{D \setminus E_n} \frac{\mu}{\mu_n} dxdt \int_{D \setminus E_n} \mu_n f_{nxx}^2 \cdot |\sqrt{\mu} - \sqrt{\mu_n}|^2 dxdt.$$

Estimating the first factor by the use of the Bunjakovskiĭ-Schwarz inequality and (7.2.19$_2$), and the second with the aid of the mean value theorem and the definition of E_n, we find that

(7.2.39) $$(I_3'')^2 < \epsilon K^{1/2} (LT)^{1/4} \left(\int_{D \setminus E_n} \mu_n f_{nxx}^2 dxdt \right)^{1/2}.$$

Majorizing this inequality with the help of $(7.2.39_2)$ and (7.2.33), we obtain

$$0 < I_3'' < K^*\epsilon. \tag{7.2.40}$$

Here K^* is some constant independent of ϵ and δ.

Analogously we find that

$$0 < I_3' \leqq \int_{E_n} \frac{\mu}{\mu_n} dxdt \cdot \int_{E_n} \mu_n f_{nxx}^2 dxdt \cdot \sup_{(x,t)\in D} (\sqrt{\mu} - \sqrt{\mu_n})^2. \tag{7.2.41}$$

Again using the Bunjakovskiĭ-Schwarz inequality in order to estimate the first integral, and noting the inequalities (7.2.36), (7.2.33) and $(7.2.29_2)$, we obtain

$$0 < I_3' < K^{**}\delta^{1/4}, \tag{7.2.42}$$

where K^{**} is not dependent on ϵ and δ. Inserting (7.2.42) and (7.2.40) into (7.2.37), we finally obtain

$$0 < I_3 \leqq K^*\epsilon + K^{**}\delta^{1/2}. \tag{7.2.43}$$

Since K^* and K^{**} are not dependent on ϵ and δ, while ϵ and δ are arbitrarily small, this shows that

$$\lim_{n\to\infty} I_3 = 0. \tag{7.2.44}$$

Comparing (7.2.44) and (7.2.34) with $(7.2.25_0)$ proves $(7.2.24_1)$, and with it Theorem 20.

We now turn to the construction of the generalized solution to the problem. Let $u_0(x)$ and $u_i(t)$ $(i = 1, 2)$ be twice continuously differentiable for $0 \leqq x \leqq L$ and $0 \leqq t \leqq T$ respectively, and in addition let

$$u_0(0) = u_1(0); \qquad u_0(L) = u_2(0). \tag{7.2.45}$$

We obtain

$$\omega(x,h) = \begin{cases} \exp\left(-\dfrac{x^2}{h^2 - x^2}\right) & \text{for } |x| < h \\ 0 & \text{for } |x| \geqq h; \end{cases} \tag{$7.2.46_1$}$$

$$\kappa = \int_{-\infty}^{\infty} \omega(s, 1)\, ds, \tag{$7.2.46_2$}$$

and we define $a_n(u)$ as the average of $a(u)$ with kernel ω, i.e. we set

$$a_n(u) = \frac{n}{\kappa}\int_{-\infty}^{\infty} \omega\left(u - s, \frac{1}{n}\right) a(s)\, ds. \tag{7.2.47}$$

Since $a(u)$ is continuous everywhere except at the points $u = s_i$ $(i = 1, \cdots, N)$, it is obvious that $a_n(u)$ converges uniformly to $a(u)$ on any closed set not containing a point s_i. From $(7.2.5_2)$ it follows that

$$\dot{a}_n(u) \geqq c_0 > 0. \tag{7.2.48}$$

We now define the u_n as the solution of the boundary problem

$$\frac{\partial^2 u_n}{\partial x^2} = \frac{\partial a_n(u_n)}{\partial t}; \qquad (x,t) \in D; \tag{7.2.49$_1$}$$

$$u_n|_{t=0} = u_0(x); \quad u_n|_{x=0} = u_1(t); \quad u_n|_{x=L} = u_2(t) \tag{7.2.49$_2$}$$

in the domain D. The existence and uniqueness of u_n follows, for example, from a result of M. Gevrey [**50**]. Here $\partial u_n/\partial t$ and $\partial^2 u_n/\partial x^2$ satisfy a Hölder condition with nonzero index with respect to x and t. We have

THEOREM 21. *The sequence $\{u_n\}$ converges uniformly to the generalized solution of the Stefan problem.*

In fact, from the maximum principle it follows that $\{u_n\}$ is uniformly bounded by constants which majorize the moduli of u_0, u_1 and u_2. We estimate the derivative of u_n. Let $V(x,t)$ be twice continuously differentiable in $\overline{D}$ with respect to x and once with respect to t; moreover, let

$$V(0,t) = \dot{u}_1(t); \qquad V(L,t) = \dot{u}_2(t). \tag{7.2.50}$$

Multiplying $(7.2.49_1)$ by $u_{n_t} - V$, integrating on the domain

$$D_\tau = \{0 < x < L;\ 0 < t < \tau \leqq T\}, \tag{7.2.51}$$

carrying out an integration by parts and using conditions (7.2.50) and $(7.2.49_2)$, we obtain

$$\begin{aligned}
&\int_{D_\tau} \dot{a}_n(u_n)\left(\frac{\partial u_n}{\partial t}\right)^2 dx\,dt + \frac{1}{2}\int_0^L \left(\frac{\partial u_n}{\partial x}\right)^2 \Bigg|_{t=\tau} dx \\
&= \int_0^L a_n(u_n)\, V\Big|_{t=0}^{\tau} dx - \int_{D_\tau} a_n \frac{\partial V}{\partial t}\, dx\,dt + \frac{1}{2}\int_0^L \dot{u}_0^2(x)\, dx \\
&\quad + \int_0^\tau u_n \frac{\partial V}{\partial x}\Big|_{x=0}^{L} dt - \int_D u_n \frac{\partial^2 V}{\partial x^2}\, dx\,dt.
\end{aligned} \tag{7.2.52}$$

Since $\{u_n\}$ is uniformly bounded, $\{a_n(u_n)\}$ is also. But this proves that the right side of (7.2.52) is uniformly bounded with respect to n (for $n = 1, 2, \cdots,$) as well as with respect to τ (for $0 < \tau \leqq T$). Since in addition $a_n(u_n) \geqq c_0 > 0$, this implies that the integrals

$$\int_0^L dx \int_0^\tau \left(\frac{\partial u_n}{\partial t}\right)^2 dt; \quad \int_0^L \left(\frac{\partial u_n}{\partial x}\right)^2 dx$$

are uniformly bounded with respect to n and τ. Hence from the uniform boundedness of $\{|u_n|\}$ it follows that $\{u_n\}$ is uniformly bounded and equicontinuous. Consequently it is possible to obtain a uniformly convergent subsequence from $\{u_n\}$. Let

$$\lim_{n_j \to \infty} u_{n_j} = u(x,t). \tag{7.2.53}$$

We now show that $u(x,t)$ is a generalized solution of the Stefan problem. In fact, we will show that $\{u_n\}$ coincides with the chosen subsequence. Let us choose any $F(x,t)$ belonging to the class of functions originally used in the definition of a generalized solution; we multiply equation $(7.2.49_1)$ by F and integrate by parts, obtaining

$$\int_D (a_n(u_n) F_t + u_n F_{xx})\, dxdt + \int_0^L F(x,0)\, a_n(u_0(x))\, dx$$

$$+ \int_0^T [F_x(0,t)\, u_1(t) - F_x(L,t)\, u_2(t)]\, dt = 0. \tag{7.2.54}$$

From the uniform boundedness of $\{a_n(u_n)\}$ there exists a subsequence $\{u_n'\}$ such that the subsequence $\{a_n'(u_n')\}$ corresponding to it is weakly convergent to a measurable bounded function $A(x,t)$. However, as we noted above, on any closed set not containing points $u = s_i$ $(i = 1, \cdots, N)$ the sequence $\{a_n(u_n)\}$ converges uniformly to $a(u)$. Consequently

$$A(x,t) = a[u(x,t)] \quad \text{for} \quad u(x,t) \neq s_i. \tag{7.2.55}$$

At the points $u = s_i$ we determine $a(u)$ by the equation

$$a(s_i) = A(x,t). \tag{7.2.56}$$

Passing to the limit of the chosen subsequence in (7.2.54) yields the identity (7.2.8). Here the convergence of $A(x,0)$ to $a_0(x)$ is guaranteed by the construction; $a_0(x)$ is assumed to be preassigned in correspondence to the condition (7.2.4).

It is possible to prove that the limit function satisfies the condition

$$a(s_i - 0) < a(s_i) < a(s_i + 0). \tag{7.2.57}$$

This shows that $u(x,t)$, and correspondingly $a[u(x,t)]$, satisfies every condition for the generalized solution of the Stefan problem, as was to be proved for the existence theorem. Thus the theorem is proved.

REMARK. From the uniqueness of the generalized solution we see that $\{u_n\}$ converges to u. Thus the generalized solution is determined by the algorithm.

The more general result which we will now formulate is due to O. A. Oleĭnik [107]. We seek a function $u(x,t)$, continuous in $\overline{D}$ ($D = \{0 < x < L;\ 0 < t < T\}$) such that throughout D, except on the curve $x = x_i(t)$ at which $u = s_i$ $(i = 1, \cdots, N)$, the equation

$$a(x,t,u)\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial^2 x} + b(x,t,u)\frac{\partial u}{\partial x} + c(x,t,u)\,u + f(x,t,u) \tag{7.2.58$_1$}$$

is satisfied; on the curve $x = x_i(t)$ we have the condition

$$\lambda(x,t,u)\frac{dx_i}{dt} = k(x,t,u)\frac{\partial u}{\partial x}\bigg|_{u=s_i-0} - k(x,t,u)\frac{\partial u}{\partial x}\bigg|_{u=s_i+0} \tag{7.2.58$_2$}$$

and on the boundary of D, the conditions

$$u(x,0) = u_0(x);\quad u(0,t) = u_1(t);\quad u(L,t) = u_2(t). \tag{7.2.58$_3$}$$

Here

$$a = a_i(x,t);\qquad b = b_i(x,t);\qquad c = c_i(x,t);$$

$$f = f_i(x,t);\quad \lambda = \lambda_i(x,t);\quad k = k_i(x,t) \text{ for } s_{i-1} < u < s_i \tag{7.2.58$_4$}$$

$$(s_0 = -\infty;\quad s_{N+1} = +\infty;\quad i = 1,2,3,\cdots,N+1),$$

where $a_i, b_i, c_i, f_i, \lambda_i$ and k_i are smooth functions in $\overline{D}$ of x and t, such that

$$a_i \geqq a_0 = \text{const} > 0;\quad k_i \geqq k_0 = \text{const} > 0;\quad \lambda_i \geqq 0. \tag{7.2.58$_5$}$$

We choose $c_0 = \text{const} < s_1$, and set

$$\gamma_i(x,t) = \sum_{j=1}^{i-1} \lambda_j(x,t); \tag{7.2.59}$$

in addition we define ϕ, ψ, v and H by the equations

$$\phi(x,t,u) = \int_{c_0}^{u} a(x,t,s)\, k(x,t,s)\, ds + \gamma_i(x,t) \text{ for } s_{i-1} < u < s_i;$$

$$\psi(x,t,u) = \int_{c_0}^{u} [k(x,t,s)\, b(x,t,s) - k_x(x,t,s)]\, ds;$$

$$v(x,t,u) = \int_{c_0}^{u} k(x,t,s)\, ds; \tag{7.2.60}$$

$$H(x,t,u) = \phi_t - \psi_x + kcu + kf.$$

Here $\partial h/\partial x$ is the total derivative of $h(x,t,u(x,t))$ and h_x the partial derivative of $h(x,t,u)$.

In the notation (7.2.60), equation (7.2.58$_1$) assumes the form

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial(\psi - v_x)}{\partial x} + H. \tag{7.2.61}$$

As in the particular problems investigated above, the generalized solution of the Stefan problem will be a function $u(x,t)$ which is bounded and measurable in D and satisfies the integral identity

$$\int_D \{\phi(x,t,u)\, F_t + v(x,t,u)\, F_{xx} - [\psi(x,t,u) - v_x(x,t,u)]\, F_x + H(x,t,u)\, F\}\, dxdt$$

$$+ \int_0^L \phi(0,x,u_0(x))\, F(x,0)\, dx \tag{7.2.62}$$

$$+ \int_0^T [v(x,0,u_1(t))\, F_x(0,t) - v(x,L,u_2(t))\, F_x(L,t)]\, dt = 0,$$

for an arbitrary twice continuously differentiable function $F(x,t)$ in D which vanishes for $t = T$, $x = 0$ and $x = L$. Here $\phi(x,t,u)$ is considered as a function of x and t which at every point where $u = s_i$ assumes definite values satisfying the inequality

$$\phi(x,t,s_i - 0) \leqq \phi(x,t,s_i) \leqq \phi(x,t,s_i + 0). \tag{7.2.63}$$

As in the above investigation of simpler cases, the following assertions are valid:

a) If a classical solution to problem (7.2.58$_i$) exists then it is a generalized solution.

b) The generalized solution to problem (7.2.58$_i$) is unique.

c) A generalized solution to (7.2.58$_i$) can be constructed as the limit of a uniformly convergent sequence of functions $u^n(x,t)$, arising as the solutions to the boundary problems

$$(7.2.64_1) \qquad \frac{\partial \phi^n}{\partial t} = \frac{\partial^2 v^n}{\partial x^2} + \frac{\partial (\psi^n - v_x^n)}{\partial x} + H^n; \ (x,t) \in D;$$

$$(7.2.64_2) \qquad u^n|_{t=0} = u_0^n(x); \quad u^n|_{x=0} = u_1^n(t); \quad u^n|_{x=L} = u_2^n(t),$$

where $\phi^n, \cdots, H^n$ are obtained as averages, with the infinitely differentiable kernel $\omega(u-s; 1/n)$, of the functions $\phi, \cdots, H$; u_0^n, u_1^n, and u_2^n, which converge to u_0, u_1 and u_2 uniformly together with $\dot{u}_0^n$, $\dot{u}_i^n$ and $\ddot{u}_i^n$ $(i = 1, 2)$ and satisfy the consistency conditions

$$(7.2.64_3) \qquad u_0^n(0) = u_1^n(0); \quad u_0^n(L) = u_2^n(0).$$

The notion of a generalized solution to the Stefan problem is a natural spatial extension of the problem. For definiteness let us examine the spatial problem posed by S. L. Kamenomostskaja. Let D be a domain in 3-dimensional space $\{X = (x,y,z)\}$, bounded by a triply continuously differentiable surface. Furthermore, let $\Omega = \{D + (0 < t < T)\}$ be a cylinder in the space (X,t) of height T. We examine the boundary problem: Find $u(x,t)$ continuous in $\overline{\Omega}$, satisfying in Ω the equation

$$(7.2.65_1) \qquad c(u)\frac{\partial u}{\partial t} = \Delta u; \qquad (X,t) \in \Omega$$

and boundary conditions of the form

$$(7.2.65_2) \quad u|_S = f(P); \quad u|_{t=0} = \phi(X); \quad X \in D \quad (P = (X,t)).$$

Here $c(u)$ is defined as in (7.2.2$_i$), and S denotes the lateral surface $S = \{F \times (0 < t < T)\}$. On the surfaces $\phi_i(x,t) = 0$ on which $u = s_i$, we have the Stefan conditions

$$(7.2.65_3) \quad b_i\frac{\partial \Phi_i}{\partial t} + (\operatorname{grad} u|_{u=s_i-0} - \operatorname{grad} u|_{u=s_i+0}, \ \operatorname{grad} \Phi_i) = 0.$$

The function $u(X,t)$, as defined, is the classical solution of the Stefan problem (7.2.65$_i$). The generalized solution is introduced exactly as in the one-dimensional case. Namely, we define $a(u)$ by using the substitution (7.2.4), after which we define the generalized solution of the problem (7.2.65$_i$) as a bounded, measurable function $u(x,t)$ in Ω satisfying the integral identity

$$\int_{\Omega}(u\Delta\psi + a(u)\psi_t)\,d\omega - \int_S f(P)\frac{\partial\psi}{\partial n}\,d\sigma + \int_D a_0(X)\psi(X,0)\,dX = 0, \tag{7.2.66}$$

in which $\psi(X,t)$ is an arbitrary twice continuously differentiable function which vanishes for $t = T$ and on S.

As in the one-dimensional case we show that if the classical solution exists, then it is a generalized solution. Furthermore, by exactly the same method as in the one-dimensional case, we prove the uniqueness theorem. For the proof of the existence theorem for a generalized solution, one can use the above method of smoothing of the coefficients, or, as is done by Kamenomostskaja, the method of finite differences.

We will limit ourselves to these observations, referring the reader to the original work of S. L. Kamenomostskaja for the complete details.

CHAPTER VIII

NUMERICAL METHODS FOR THE SOLUTION OF THE STEFAN PROBLEM

§1. Methods based on a reduction to a countable system of ordinary differential equations

In the Introduction we noted that the method of reduction of the Stefan problem to a countable system of ordinary differential equations was developed by V. G. Melamed [**99**] as a method convenient for numerical calculation on the EVCM. Simultaneously, in order to justify the method we proved existence and uniqueness of a solution to problem B_1. We now present Melamed's method in a somewhat modified form, approaching it by using the heat potential employed in Chapter I. For the sake of simplicity we limit our examination to a single-phase Stefan problem of the form

$$(8.1.1_1) \qquad Lu \equiv u_{xx} - u_t = 0; \quad 0 < x < y(t); \quad 0 < t < T;$$

$$(8.1.1_2) \qquad \begin{aligned} &u|_{x=0} = f(t) < 0; \quad u|_{x=y(t)} = 0; \quad 0 < t < T; \\ &u|_{t=0} = \phi(x) < 0; \quad 0 < x < l; \end{aligned}$$

$$(8.1.1_3) \qquad \dot{y}(t) = \frac{\partial}{\partial x} u(y(t), t); \quad y(0) = l > 0.$$

We assume that f and ϕ are respectively twice and three times continuously differentiable functions, satisfying the conditions

$$(8.1.1_4) \qquad f(0) = \phi(0); \qquad \phi(l) = 0.$$

Thus we will examine problem B_1 (see Chapter II, §1) with $u_2 \equiv 0$. We set

$$z = \frac{x}{y(t)}; \quad \tau = \int_0^t \frac{d\lambda}{y^2(\lambda)}; \quad Z(\tau) = y(t);$$

$$(8.1.2) \qquad \Phi(z) = \phi(lz) - f(0)(1 - z);$$

$$U(z, \tau) = u(x, t) - f(t)(1 - z).$$

Let us take z and τ as independent variables and find U, Z and t, extending the number of unknown functions from two to three. In the variables z, τ, U, Z, t, problem $(8.1.1_i)$ $(i = 1,2,3)$ is written in the form

$$(8.1.3_1)\quad LU = -z\frac{d\ln Z}{d\tau}\frac{\partial U}{\partial z} + f(t)(1-z)Z^2(\tau) + zf(t)\frac{d\ln Z}{d\tau};\quad 0<z<1;$$

$$(8.1.3_2)\quad U|_{z=0} = U|_{z=1} = 0;\quad U|_{\tau=0} = \Phi(z);\quad 0<z<1;$$

$$(8.1.3_3)\quad \frac{dZ}{d\tau} = Z(\tau)\frac{\partial U}{\partial z}\bigg|_{z=1} + Z(\tau)f(t);\quad Z(0) = l;$$

$$(8.1.3_4)\quad \frac{dt}{d\tau} = Z^2(\tau);\quad t(0) = 0.$$

Let $g(x, \zeta, \tau - \tau_1)$ be the Green's function of the first boundary problem for the equation $LU = 0$ on the interval $0 \leqq z \leqq 1$. From $(8.1.3_1)$ and $(8.1.3_2)$ it follows that

$$(8.1.4_1)\quad \begin{aligned} U(z,\tau) = &\int_0^1 \Phi(\zeta)\,g(z,\zeta,\tau)\,d\zeta \\ &+ \int_0^\tau d\tau_1 \int_0^1 \Psi(\zeta,\tau_1)\,g(z,\zeta,\tau-\tau_1)\,d\zeta, \end{aligned}$$

where

$$(8.1.4_2)\quad \begin{aligned} \Psi(\zeta,\tau_1) = &\zeta\frac{d\ln Z(\tau_1)}{d\tau_1}\frac{\partial}{\partial z}U(\zeta,\tau_1) \\ &- f[t(\tau_1)](1-\zeta)Z^2(\tau_1) - \zeta f[(\tau_1)]\frac{d\ln Z(\tau_1)}{d\tau_1}. \end{aligned}$$

The equivalence of the system $(8.1.4_1)$, $(8.1.3_i)$ $(i = 2,3,4)$ to system $(8.1.1_i)$ $(i = 1,2,3)$ is obvious. Since the solution to problem B_1 exists and is unique, there exists a unique solution to the system $(8.1.4_i)$ $(i = 1,2)$, $(8.1.3_4)$.

We now note that [179]

$$(8.1.5)\quad g(z,\zeta,\tau) = 2\sum_{n=1}^{\infty}\exp(-n^2\pi^2\tau)\sin n\pi z\cdot\sin n\pi\zeta.$$

Consequently $U(z,\tau)$ can be represented in the form

$$(8.1.6)\quad U(z,\tau) = \sum_{n=1}^{\infty} A_n(\tau)\sin n\pi z,$$

where, moreover, from $(8.1.4_1)$ and from the boundedness of $\partial U/\partial z$ for $0 \leqq z \leqq 1$, it obviously follows that the series (8.1.6) can be differentiated termwise.

Inserting (8.1.6) into $(8.1.4_1)$ and comparing the coefficients of $\sin n\pi z$ in corresponding terms of the series, we find that

$$A_n(\tau) = a_n(\tau) + \sum_{m=1}^{\infty} m\pi \int_0^{\tau} d\tau_1 \int_0^1 \zeta \frac{d\ln Z}{d\tau_1} \cos m\pi\zeta \cdot \sin n\pi\zeta \cdot \exp[-n^2\pi^2(\tau - \tau_1)] \cdot A_m(\tau_1)\, d\zeta, \tag{8.1.7$_1$}$$

where

$$a_n(\tau) = \int_0^1 \Phi(\zeta) \exp(-n^2\pi^2\tau) \sin n\pi\zeta d\zeta - \int_0^{\tau} \exp[-n^2\pi^2(\tau - \tau_1)] d\tau_1 \cdot \int_0^1 \left\{ f[t(\tau_1)](1-\zeta) Z^2(\tau_1) + \zeta f[t(\tau_1)] \frac{d\ln Z}{d\tau_1} \right\} \sin n\pi\zeta \cdot d\zeta. \tag{8.1.7$_2$}$$

The system $(8.1.7_1)$ is obviously equivalent to the following countable system of ordinary differential equations:

$$\exp(-n^2\pi^2\tau) \frac{d}{d\tau} [\exp(n^2\pi^2\tau)(A_n(\tau) - a_n(\tau))] = \sum_{m=1}^{\infty} m\pi \frac{d\ln Z}{d\tau} A_m(\tau) \int_0^1 \zeta \cos m\pi\zeta \sin n\pi\zeta \cdot d\zeta. \tag{8.1.7$_3$}$$

We note further that from (8.1.6) and $(8.1.3_3)$ it follows that

$$\frac{d\ln Z}{d\tau} = f[t(\tau)] + \sum_{m=1}^{\infty} m\pi(-1)^m A_m(\tau). \tag{8.1.7$_4$}$$

Thus to obtain the solution of the original problem it sufficies to determine $A_n(\tau)$ $(n = 1, 2, \cdots)$, Z and t from the system $(8.1.7_i)$ $(i = 3, 4)$, $(8.1.3_4)$. The existence of a solution to this system follows from its derivation. The uniqueness of the solution easily follows from the uniqueness of the solution to problem B_1, proved in Chapter I, as well as from the obvious uniqueness of a solution to the truncated system obtained from the system $(8.1.7_3)$—$(8.1.3_4)$ if we take $A_m(\tau) \equiv 0$ for $m > N$, where $N > 0$ is an arbitrary fixed number.

It is obvious that by using (8.1.2) in the resulting system of ordinary differential equations it is easy to return to the original variables x, t, y, u. Without dwelling on details, we note that in order to obtain the system at once, without resorting to the construction of the solution to the problem by the use of the Green's function, we may simply assume that for any fixed t the function $u(x,t)$ can be expanded in the form of a Fourier series in $\sin[n\pi x/y(t)]$, which can be differentiated termwise twice with respect to x and once with respect to t. This method is used in the work of V. G. Melamed.

§2. Difference methods for the solution of a problem of Stefan type

Introduction. For the solution of a problem of Stefan type one may propose several different finite difference schemes. The proposed schemes can be split into the following groups.

A. *Schemes with variable time steps, in which the moving boundary is located on the space nodes of the mesh.* Schemes of this type are designed for the solution of two-phase and one-phase Stefan problems in the case when the conditions of the problem guarantee the monotonic variation of the interphase boundary. The first scheme of this type was developed by J. Douglas and T. M. Gallie [32] for the solution of the simplest model of a single-phase Stefan problem. The time mesh is chosen so that the moving boundary is guaranteed to be located on one node of the mesh as one proceeds from one time step to the next; to determine the time step Douglas and Gallie construct an iterative process, using an integral equation of Evans-Kyner (see Chapter VII, §1) equivalent to the Stefan condition.

Later, in the works of F. P. Vasil'ev and A. B. Uspenskiĭ and in their joint work with B. M. Budak [15] the method of Douglas and Gallie was extended, with a small modification of the scheme, to considerably more general one- and two-phase problems. In the work of Douglas and Gallie the convergence of the process to the solution of the original boundary problem was proved, together with stability of the difference scheme. Budak and his coauthors limited themselves to proving convergence, and did not solve the problem of stability. In all these articles essential use was made of the fact that boundary conditions of the second kind were prescribed on known boundaries.

B. *Schemes with mesh fixed in time which use fractional space steps to determine the position of the phase boundary.* Schemes of this type

were described in the above-mentioned works of Douglas and Gallie, Budak and his coauthors. In the latter work a similar scheme was applied to the construction of the solution ot the Stefan problem with an arbitrary number of nonmonotonically moving boundaries. The detailed investigation of the problem was announced by them. Our attempts to use similar schemes for the solution of a problem with nonmonotonically moving boundary (the problem of crystallization of a melt for insertion of a plate in it, investigated in Chapter V) have been unsuccessful. It is probable that the application of a scheme of this type for solving a problem with nonmonotonically moving boundary requires great caution.

C. *Schemes with a fixed number of space nodes.* Schemes of this type are proposed in two forms. In the first form passage to the finite difference approximation of the differential equation of the problem is preceded by an affine transformation of the domain of existence of each phase into an interval of fixed length. In the second a mesh is used with nodes situated on lines obtained by a parallel translation of the boundary. A scheme of the first type was first described by M. Lotkin [96] and applied to a one-phase problem, similar to the problem of Kyner but with a nondegenerate domain of initial existence for each phase. Later an analogous scheme was described by Budak and his collaborators [15] for determining the solution to the Stefan problem with an arbitrary finite number of nonmonotonically moving boundaries. In the same article a strict justification of a similar algorithm was announced.

A scheme of the second type was introduced by W. D. Murray and F. Landis [104] without a strict justification. It is probable that this can be applied to the solution of the problem with degeneration of the domain of existence of one of the phases at the initial time, which is in principle lacking for schemes of the first type. However, this problem requires special work.

D. *Schemes for the construction of the generalized solution of the Stefan problem.* In a purely theoretical direction, as a means for proving existence of the generalized solution to the Stefan problem, S. L. Kamenomostskaja has studied an explicit difference scheme [73]. The basic result, from a computing point of view, obtained by her is that explicit difference schemes for constructing the generalized solution of the Stefan problem are stable if the mesh length in time is of the order of the square of the length of the space mesh. Numerically, the criterion of stability

turns out to be similar to the well-known criterion of stability of the explicit difference scheme for the Fourier equation with constant coefficients. Practical tests of this method show it to be not very effective.

The construction of the generalized solution to the Stefan problem with the use of the method of smoothing coefficients, introduced by O. A. Oleĭnik (see Chapter VII, §2) can be realized with the help of any difference scheme sutiable for the solution of the boundary problem for a quasilinear equation of parabolic type with smooth coefficients. The essential element of this method is the smoothing of the function $a(u)$ of (7.2.6). Practical recommendations for this method are contained in the article [15]. The same author has announced more details illuminating the problem. Some data on the use of this method for the problem of zonal noncrucible melting are given in Supplement 3, §6.

We now pass to a more detailed description of the above schemes.

A. **Schemes with varying time steps, in which the moving boundary is displaced by one space node of the mesh.** We again examine the one-phase Stefan problem in the formulation of Kyner (see Chapter VII, §1):

$$\frac{\partial}{\partial x}\left(p\frac{\partial u}{\partial x}\right) = f\frac{\partial u}{\partial t};\ (x,t) \in D = \{0 < x < y(t), t > 0\};$$

$$(8.2.1_1)\quad p\frac{\partial u}{\partial x} = -a;\quad x = 0;\quad t > 0;$$

$$u = 0;\quad x = y(t);\quad t > 0;\quad y(0) = 0;$$

$$(8.2.1_2)\quad \dot{y}(t) = b - pu_x;\quad x = y(t);\quad t > 0;\quad b = \text{const} > 0.$$

Here a, f and p are defined by the conditions (7.1.2), (7.1.4$_1$)—(7.1.4$_4$). We recall that if condition (8.2.1$_1$) is valid, then condition (8.2.1$_2$) is equivalent to condition (7.1.8), which can be written in the form

$$(8.2.1_3)\quad t = \frac{1}{b}\left\{y(t) - \int_0^t a(s)\,ds + \int_0^{y(t)} F(x,t,u(x,t))\,dx + \int_0^t d\tau \int_0^{y(\tau)} H(x,\tau,u(x,\tau))\,dx\right\}.$$

Here F and H are defined by the equations

$$(8.2.1_4)\quad F(x,t,u) = \int_0^u f(x,t,s)\,ds;\quad H(x,t,u) = -\int_0^u f_t(x,t,s)\,ds.$$

We choose a fixed $h > 0$ and some sequence $\tau_k > 0$ which is subject to

later choice. We set

$$x_i = ih; \quad t_m = \sum_{k=0}^{m} \tau_k; \quad \tau_0 = 0; \quad f_{ij} = f(x_i, t_j) \tag{8.2.2}$$

for any function $f(x,t)$.

Let D_h be the mesh domain

$$D_h = \{xi;\ t_j;\ i = 1,2,3,\cdots,\ j-1;\ j = 1,2,3,\cdots\}; \tag{8.2.3$_1$}$$

$\overline{D}_h$ the closure of D_h obtained by the addition of the nodal lines

$$\Gamma_h^0 = \{0;\ t_j;\ j = 0,1,2,\cdots\}; \quad \Gamma_h^1 = \{x_j;\ t_j;\ j = 0,1,2,\cdots\}; \tag{8.2.3$_2$}$$

$D_{h,m}$ ($\overline{D}_{h,m}$) the cross-section of D_h ($\overline{D}_h$) at the plane $t = t_m$, and D_h^m ($\overline{D}_h^m$) the intersection of D_h ($\overline{D}_h$) with the line $t \leqq t_m$. Further, let $v(x,t)$ be an arbitrary function defined in $\overline{D}_h$. We set

$$v_x = \frac{v(x+h,t) - v(x,t)}{h}; \quad v_{\bar{x}} = \frac{v(x-h,t) - v(x,t)}{-h};$$
$$v_t = \frac{v(x,t+\tau') - v(x,t)}{\tau'}; \quad v_{\bar{t}} = \frac{v(x,t) - v(x,t-\tau)}{\tau}, \tag{8.2.4}$$

where

$$\tau = \tau_m; \quad \tau' = \tau_{m+1} \text{ for } t = t_m. \tag{8.2.4*}$$

The system (8.2.1$_1$), (8.2.1$_3$) consists of a system of finite difference equations on $\overline{D}_h$; we choose the sequence $\{\tau_k\}$ such that the points $(x = y(t_k), t_k)$ belong to Γ_h^1. For the integrals in (8.2.1$_3$) we use a quadrature formula, with the points of D_h chosen as nodal points. For definiteness, we can use the rectangle formula. Then the system of algebraic equations corresponding to the system (8.2.1$_1$), (8.2.1$_3$) can be written in the form.

$$(1/2)\left[(pv_x)_{\bar{x}} + (pv_{\bar{x}})_x\right] = fv_{\bar{t}}, \quad (x,t) \in D_h; \tag{8.2.5$_1$}$$

$$pv_x = -a, \quad (x,t) \in \Gamma_h^0; \quad v = 0, \quad (x,t) \in \Gamma_h^1; \tag{8.2.5$_2$}$$

$$t_m = \frac{1}{b}\left\{ y_m - \sum_{k=1}^{m} a_k\tau_k + h\sum_{k=1}^{m} F(x_k, t_m, v_{k,m}) + h\sum_{k=1}^{m}\tau_k \sum_{l=1}^{k} H(x_l, t_k, v_{l,k}) \right\}; \tag{8.2.5$_3$}$$

$$t_m = \sum_{j=0}^{m} \tau_j; \quad \tau_0 = 0; \quad y_m = mh. \tag{8.2.5$_4$}$$

The solution of system $(8.2.5_i)$ $(i = 1, 2, 3)$ with the choice of the mesh τ_k ensuring the compatibility of the solution of this system with the conditions $(8.2.5_4)$, is realized by iteration. Namely, let us assume that $t_k > 0$ $(k = 1, \cdots, m)$ is found and $v(x, t)$ is defined in $\overline{D}_h^m$. We choose an arbitrary $\tau_{m+1}^0 > 0$ and set

$$t_{m+1}^0 = t_m + \tau_{m+1}^0; \quad v^0(x_i, t_{m+1}^0) = v(x, t_m)$$

$$(i = 0, 1, 2, \cdots, m); \tag{8.2.6}$$

$$v^0(x_{m+1}, t_{m+1}^0) = 0.$$

We assume further, that $v^k(x, t_{m+1}^k)$ and $t_{m+1}^k = t_m + \tau_{m+1}^k$ is defined for $k = 0, \cdots, n$ and $x/h = 0, \cdots, m+1$. We then define $v^{k+1}(x, t_{m+1}^{k+1})$ and τ_{m+1}^{k+1} by the conditions

$$(1/2)\left[(Pv_x^{k+1})_{\bar{x}} + (Pv_{\bar{x}}^{k+1})_x\right] = \Phi v_{\bar{t}}^{k+1}; \quad (x, t) \in D_{h,m+1}^{k+1};$$

$$Pv_x^{k+1} = -a; \quad x = 0; \quad t = t_{m+1}^{k+1}; \quad v^{k+1} = 0;$$

$$x = (m+1)h; \quad t = t_{m+1}^{k+1};$$

$$t_{m+1}^{k+1} = \frac{1}{b}\left\{ y_{m+1} - \sum_{k=1}^{m} a_k \tau_m - a_{m+1}\tau_{m+1}^k \right.$$

$$+ h\sum_{j=1}^{m+1} F(x_j, t_{m+1}^k, v_{j,m+1}^k) + h\sum_{k=1}^{m} \tau_k \sum_{j=1}^{k} H(x_j, t_k, v_{j,k}) \tag{8.2.7}$$

$$\left. + h\tau_{m+1}^k \sum_{l=1}^{m+1} H(x_l, t_{m+1}^k, v_{l,m+1}^k) \right\};$$

$$\tau_{m+1}^{k+1} = t_{m+1}^{k+1} - t_m; \quad v^{k+1}(x, t) = v(x, t) \quad \text{for} \quad (x, t) \in D_h^m.$$

Here

$$P = p(x, t^k, v^k(x, t^k)); \quad \Phi = f(x, t^k, v^k(x, t^k)). \tag{8.2.7*}$$

In (8.2.7) $D_{h,m+1}^{k+1}$ is defined exactly as $D_{h,m}$ but with replacement of t_{m+1} by t_{m+1}^{k+1}.

Assuming convergence of this algorithm, we make use of the fact that

$$\tau_{m+1} = \lim_{k \to \infty} \tau_{m+1}^k; \quad t_{m+1} = t_m + \tau_{m+1};$$

$$v(x, t_{m+1}) = \lim_{k \to \infty} v^k(x, t_{m+1}^k), \tag{8.2.8}$$

thereby completing the construction of the numerical algorithm.

The proof of the legitimacy of the algorithm is found in the work of Douglas and Gallie for the particular case

(8.2.9) $$a \equiv 0; \quad b = 1; \quad p \equiv 1; \quad f \equiv 1.$$

In this case they prove the following theorem.

THEOREM 22 (DOUGLAS AND GALLIE). a) *The limits*

(8.2.10) $$\lim_{k \to \infty} \tau_m^k = \tau_m; \quad \lim_{k \to \infty} v^k = v \quad (m = 0, 1, 2, \cdots)$$

exist, and moreover the convergence is monotone.

b) *Let* t_m, $v_{i,m}$ $(i = 0, \cdots, m;\ m = 0, 1, \cdots)$ *be the solution of problem* $(8.2.5_1) - (8.2.5_4)$. *Construct the polygon* $x = y_h(t)$ *with vertices at the points* (mh, t_m) $(m = 0, 1, \cdots)$. *Let* $t = t_h(x)$ *be the function inverse to* $x = y_h(t)$. *Then for any* $\epsilon > 0$ *on the domain* $0 \leqq x \leqq 1 - \epsilon$ *the limit*

(8.2.11) $$\lim_{h \to 0} t_h(x) = T(x)$$

exists, and moreover $T(x)$ *is monotone and satisfies a Lipschitz condition:*

(8.2.12) $$x_2 - x_1 \leqq T(x_2) - T(x_1) \leqq \frac{x_2 - x_1}{\epsilon} \quad \text{for} \quad 0 \leqq x_1 \leqq x_2 \leqq 1 - \epsilon.$$

c) *Let* $x = y(t)$ *be a function inverse to* $t = T(x)$. *Moreover, let* $u(x, t)$ *be the solution to problem* $(8.2.1_3)$, *if in it we take the curve* $y(t)$ *as the boundary of the domain* D. *Then* $u(x, t), y(t)$ *satisfies equation* $(8.2.1_3)$ *and consequently* $(8.2.1_2)$.

d) *For any closed subset* F *of the domain* $D = \{0 < x < y(t);\ 0 < t \leqq T(1 - \epsilon)\}$ *and for any* $\delta > 0$ *there exists an* $\eta > 0$ *such that if* $h < \eta$ *and* F_h *is the intersection of* F *with the mesh domain* D_h, *then*

(8.2.13) $$|v(x, t) - u(x, t)| < \delta \quad \text{for} \quad (x, t) \in F_h = F \cap \overline{D}_h.$$

e) *Let there be given a fixed* $h > 0$ *and* $\epsilon > 0$. *Furthermore, let* $v_{i,m}$, t_m *be the solution of the problem* $(8.2.5_1) - (8.2.5_4)$, *constructed for* $0 \leqq x \leqq 1 - \epsilon$ *and chosen* h. *Finally, let* $v_{i,m}$, t_m^* *be the calculated approximations to the values* $v_{i,m}$, t_m *whose error depends on errors which are made for* $t = t_p$, $p \leqq m$. *Set*

(8.2.14) $$\delta_{i,m} = v_{i,m}^* - v_{i,m}; \qquad \delta = \max_{0 \leqq i \leqq p} |\delta_{i,p}|.$$

Then for arbitrary $n \geqq p$

(8.2.15) $$\max_{0 \leqq i \leqq n} |\delta_{i,n}| < \left(\frac{2}{\epsilon}\right)^{(1-\epsilon)/h} \cdot \delta; \quad |t_n - t^*| < \left(\frac{2}{\epsilon}\right)^{(1-\epsilon)/h} \cdot \delta,$$

which signifies stability of the difference scheme, since h and ϵ are constants.

A somewhat modified version of the method of variation of the time step is introduced in the work of Budak and his collaborators for the one- and two-phase cases. The computing schemes for these two cases are completely analogous. For this reason we limit ourselves to formulating their results in connection with the one-phase problem.

We examine the problem of defining $u(x,t)$ and $y(t)$ by the conditions

$$
\begin{aligned}
&a(x,t,u,p,r)\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}; \quad (x,t) \in D = \{0 < x < y(t); \quad t > 0\};\\
&\left.\frac{\partial u}{\partial x}\right|_{x=0} = -q(t,u); \quad u|_{x=y(t)} = 0; \quad t > 0;\\
&\gamma(x,t,u,p)\dot{y}(t) = -\frac{\partial u}{\partial x} + \Phi(x,t,u,p) \quad \text{for} \quad x = y(t); \quad t > 0;\\
&y(0) = 0.
\end{aligned}
\tag{8.2.16}
$$

Here

$$
p = \frac{\partial u}{\partial x}; \qquad r = \frac{\partial^2 u}{\partial x^2}.
\tag{8.2.16*}
$$

The system (8.2.16) corresponds to the system of finite difference equations

$$
\begin{aligned}
&\alpha v_{x\bar{x}} = v_{\bar{t}}, \quad (x,t) \in D_h;\\
&v_x = -\rho, \quad (x,t) \in \Gamma_h^0; \quad v = 0, \quad (x,t) \in \Gamma_h^1;\\
&\lambda\frac{h}{\tau} = -v_{\bar{x}} + \psi, \quad (x,t) \in \Gamma_h^1, \quad t > 0;
\end{aligned}
\tag{8.2.17}
$$

$$
y_m = mh; \quad t_m = \sum_{k=0}^{m} \tau_k; \quad \tau_0 = 0 \quad (m = 0,1,2,3,\cdots).
$$

Here d_h, Γ_h^0, Γ_h^1 and τ are defined as above, and

$$
\begin{aligned}
&\alpha = a(x,t-\tau,v(x,t-\tau),v_x(x,t-\tau),v_{x\bar{x}}(x,t-\tau));\\
&\rho = q(t-\tau,v(0,t-\tau));\\
&\psi = \Phi(x,t-\tau,v(x,t-\tau),v_{\bar{x}}(x,t-\tau));\\
&\lambda = \gamma(x,t-\tau,v(x,t-\tau),v_{\bar{x}}(x,t-\tau)).
\end{aligned}
\tag{8.2.17*}
$$

The iteration process defining t_m and $v(x,t)$ for $(x,t) \in D_h^m$ is given by

the equations

$$
\begin{aligned}
&\alpha v_{\bar{x}x}^{s+1} = v_{\bar{t}}^{s+1}, \quad (x,t) \in D_h^{m,s+1};\\
&v_x^{s+1} = -\rho, \quad (x,t) \in \Gamma_{h,m}^{0,s+1}; \quad v^{s+1} = 0, \quad (x,t) \in \Gamma_{h,m}^{1,s+1}\\
&\tau^{s+1} = \frac{h\lambda + \tau^s(\rho + v_{\bar{x}}^s)}{\rho + \Phi}, \quad (x,t) \in \Gamma_{h,m+1}^{1,s+1};\\
&t_{m+1}^{s+1} = t_m + \tau_{m+1}^{s+1}; \quad v = \lim_{s\to\infty} v^s; \quad t_m = \lim_{s\to\infty} t_m^s.
\end{aligned}
\tag{8.2.18}
$$

Here we use the same notation as above. In their work Budak and his collaborators prove the following theorems.

THEOREM 23. *Let there be given the equations*

$$
\begin{aligned}
&q(t,u) > 0; \quad \frac{\partial q}{\partial t} \geqq 0;\\
&\frac{\partial q}{\partial u} \geqq 0; \quad Q = \max q \ \text{ for } \ t \geqq 0; \quad u \geqq 0;\\
&a(x,t,u,p,r) \geqq a_0 = \text{const} > 0 \ \text{ for } \ (x,t,u,p,r) \in G;\\
&G = \{x \geqq 0; \quad t \geqq 0; \quad u \geqq 0; \quad p \geqq -Q; \quad r \geqq 0\};\\
&\gamma(x,t,0,p) \geqq \gamma_0 = \text{const} > 0;\\
&q(t,u) + \Phi(x,t,0,p) > 0 \ \text{ for } \ x \geqq 0;\\
&t \geqq 0; \quad 0 \geqq p \geqq -Q.
\end{aligned}
\tag{8.2.19}
$$

Then for any given τ_n^0 *the iterates* $t_n^s, v^s(x,t_n^s)$ *are monotone sequences converging to the solutions* t_n, $v(x,t_n)$ *of system* (8.2.17). *Furthermore,*

$$
\begin{aligned}
&0 < v(x_i,t_n) \leqq nhQ; \quad -\rho \leqq v_x < 0;\\
&v_{\bar{x}x} \geqq 0; \quad v_{\bar{t}} > 0 \ \text{ for } \ (x,t) \in D_{h,n};\\
&\tau \geqq \frac{h\lambda}{\rho + \psi} > 0; \quad \tau = \tau_n \ \text{ for } \ t = t_n.
\end{aligned}
\tag{8.2.20}
$$

THEOREM 24. *Suppose that*

$$
\begin{aligned}
&q \equiv \text{const} > 0; \quad \frac{\partial a}{\partial r} \equiv \frac{\partial a}{\partial p} \equiv 0; \quad a \geqq a_0 > 0;\\
&\frac{\partial a}{\partial t} \leqq 0; \quad \frac{\partial a}{\partial u} \leqq 0;
\end{aligned}
\tag{8.2.21}
$$

$$\begin{aligned} &a \in C^1 \ \textit{in} \ G = \{0 \leqq x \leqq l; \quad t \geqq 0; \quad 0 \leqq u \leqq lq\}; \\ &a \in C^4 \ \textit{with respect to} \ (x, u) \ \textit{for} \ (x, t, u) \in G. \end{aligned} \tag{8.2.22}$$

$$\begin{aligned} &\frac{\partial \Phi}{\partial p} \equiv 0; \quad \Phi|_{u=0} \in C \ \textit{with respect to} \ (x, t); \\ &\frac{\partial \gamma}{\partial p} \equiv 0; \quad \gamma|_{u=0} \geqq \gamma_0 = \text{const} > 0; \quad \gamma|_{u=0} \in C^1 \ \textit{for} \ t \geqq 0. \end{aligned} \tag{8.2.23}$$

$$q + \min \Phi > l \frac{q}{a_0} \Lambda_1; \quad \Lambda_1 = \frac{q + \Phi}{\gamma}. \tag{8.2.24}$$

Furthermore, let $z(t)$ be an arbitrary curve satisfying the conditions

$$\begin{aligned} &z(0) = 0; \quad z(T) = l; \quad 0 < \Lambda_2 \leqq \frac{z(t'') - z(t')}{t'' - t'} \leqq \Lambda_1; \\ &t'', t' \in [0, T] \end{aligned} \tag{8.2.25}$$

and let $w(x, t)$ be the solution to the problem

$$\begin{aligned} &a \frac{\partial^2 w}{\partial x^2} = \frac{\partial w}{\partial t}; \quad 0 < x < z(t); \quad 0 < t \leqq T; \\ &\frac{\partial w}{\partial x} = -q; \quad x = 0; \quad t > 0; \quad w = 0; \quad x = z(t); \quad t > 0, \end{aligned} \tag{8.2.26}$$

where a and q satisfy conditions (8.2.21) *and* (8.2.22). *Assume that w has derivatives $\partial^4 w/\partial x^4$ and $\partial^2 w/\partial t^2$ satisfying the inequalities*

$$\left| \frac{\partial^4 w}{\partial x^4} \right| \leqq M(\delta); \qquad \left| \frac{\partial^2 w}{\partial t^2} \right| \leqq M(\delta) \tag{8.2.27}$$

in the domain

$$D^\delta = \{0 \leqq x \leqq z(t) - \delta; \quad t_\delta \leqq t \leqq T\}, \tag{8.2.28}$$

where δ is arbitrary, $0 < \delta < l$, and $z(t_\delta) = \delta$. Then there exists a unique solution u, y to problem (8.2.16), *which can be obtained as the limit as $h \to 0$ of solutions to problem* (8.2.17). *Furthermore,*

$$\begin{aligned} &0 \leqq u \leqq ql; \quad -q \leqq \frac{\partial u}{\partial x} \leqq 0; \quad 0 \leqq \frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial x^2} \leqq \Lambda_1 q; \\ &0 \leqq y(t) \leqq l; \quad 0 < \Lambda_2 \leqq \dot{y}(t) \leqq \Lambda_1, \end{aligned} \tag{8.2.29}$$

where Λ_2 *is some constant.*

Budak and his collaborators did not present a theorem of stability.

B. **Schemes with fixed time step which use fractional space steps to determine the phase boundary.** Schemes of this type were briefly described by Douglas and Gallie in their paper referred to above. There they investigated the problem $(8.2.1_1)$, $(8.2.1_3)$ in the case $(8.2.9)$.[1] One introduces a mesh with fixed mesh lengths $h > 0$, $\tau > 0$. Let

$$x_m = mh; \quad t_n = n\tau; \quad v_{m,n} = v(x_m, t_n) \tag{8.2.30}$$

and let y_n be the moving boundary at the moment t_n. We set

$$N_n = \mathrm{Ent}\left(\frac{y_n}{h}\right). \tag{8.2.31}$$

Then the system of difference equation analogous to the system $(8.2.5_1)$-$(8.2.5_4)$ can be written in the form

$$v_{x\bar{x}} = v_{\bar{t}}; \quad x = x_i; \quad t = t_n \ (i = 1, 2, \cdots, N_n - 1); \tag{8.2.32$_1$}$$

$$\frac{2}{y_{n+1} - h(N_n - 1)} \left\{ \frac{v_{N_n-1,n+1} - v_{N_n,n+1}}{h} - \frac{v_{N_n,n+1}}{y_{n+1} - N_n h} \right\} = \frac{v_{N_n,n+1} - v_{N_n,n}}{\tau}; \tag{8.2.32$_2$}$$

$$v_{1,n+1} - v_{0,n+1} = 0; \tag{8.2.32$_3$}$$

$$y_{n+1} = t_{n+1} - h \sum_{i=1}^{N_n - 1} v_{i,n+1} - \frac{y_{n+1} - (N_n - 1)h}{2} v_{N_n,n+1}. \tag{8.2.32$_4$}$$

In equation $(8.2.32_3)$ we insert the requirement that v vanish on the interphase boundary. In the case $N_{n+1} > N_n$, i.e. when in one time step the boundary moves more than one space node, it is necessary to evaluate $v_{i,n+2}$ supposing a linear extension of $v_{i,n+1}$ to the interval $[N_n h, y_{n+1}]$. For the solution of the system $(8.2.32_1)$—$(8.2.32_4)$ the author suggests an iteration process, completely analogous to the iteration process described for the scheme with variable time step.

The scheme used by Douglas and Gallie is essentially dependent on the monotonicity of the unknown boundary. Budak and his collaborators

[1] Strictly speaking, the problem examined by Douglas and Gallie is obtained from problem $(8.2.1_i)$ $(i = 1, 3)$, $(8.2.9)$ with the substitution $w = u - x$.

describe a scheme which does not require monotonicity of the interphase boundary. For example, let us examine the problem

$$\frac{\partial^2 u_1}{\partial x^2} = \frac{\partial u_1}{\partial t};\quad 0 < x < y(t);\quad a^2\frac{\partial^2 u_2}{\partial x^2} = \frac{\partial u_2}{\partial t};\quad y(t) < x < 1;\quad t > 0;$$

$$\frac{\partial u_1}{\partial x} = -q_1(t);\quad x = 0;\quad \frac{\partial u_2}{\partial x} = -q_2(t);\quad x = 1;\quad t > 0;$$

(8.2.33)

$$u_2(x,0) = \phi(x);\quad 0 < x < 1;\quad u_i(y(t),t) = 0;\quad i = 1,2;\quad t > 0;$$

$$\dot{y}(t) = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial x};\quad x = y(t);\quad t > 0;\quad y(0) = 0.$$

We replace the boundary $x = y(t)$ by a polygonal curve with vertices at the lines $t = m\tau$, $\tau > 0$ $(m = 0,1,\cdots)$. We introduce a mesh with nodes at the points $t_m = m\tau$, $x_i = ih$. The time step is chosen so small that the projection of each line of the polygon on the x-axis is not greater than h. Moreover, let y_m be the location of the boundary at time t_m and let N_m be defined in agreement with (8.2.31). We denote by $D_{i,h,\tau}$ the mesh domain

$$D_{1,h,\tau} = \{x_i; t_j;\ i = 1,2,\cdots,N_j - 2;\ j = 2,3,\cdots\};$$

(8.2.34)

$$D_{2,h,\tau} = \{x_i; t_j;\ i = N_j + 2, N_j + 3, \cdots, M;\ x_M = 1;\ j = 2,3,\cdots\}.$$

and select any $\alpha, \frac{1}{2} \leqq \alpha \leqq 1$. In the domain $D_{i,h,\tau}$ the author suggests a direct calculation by means of the quadratic difference scheme

$$(8.2.35_1)\qquad \alpha v_{1,\bar{x}\bar{x}}(x,t) + (1-\alpha)\, v_{1,\bar{x}\bar{x}}(x,t-\tau) = v_{1,\bar{t}}(x,t);\quad (x,t) \in D_{1,h,\tau};$$

$$(8.2.35_2)\qquad a^2[\alpha v_{2,\bar{x}\bar{x}}(x,t) + (1-\alpha)\, v_{2,\bar{x}\bar{x}}(x,t-\tau)] = v_{2,\bar{t}}(x,t);\quad (x,t) \in D_{2,h,\tau}.$$

For $(N_n - 2)h < y < y_n, t = t_n$ and $y_n < x < (N_n + 2)h, t = t_n$ the Fourier equation for u_1 and u_2 is replaced by the quadratic or four-point difference scheme with dependence on the values y_{n-1}. There are three possible cases:

$$(8.2.36)\qquad N_{n-1} = N_n;\quad N_{n-1} < N_n;\quad N_{n-1} > N_n.$$

In the first case the equation $(8.2.35_1)$ is used at the point $(N_n - 1, n)$. If $y_n = hN_n$ then at the point (N_n, n) we write the difference analog of the Stefan condition. But if $y_n > hN_n$, then at this point we write the equation $(8.2.32_2)$.

In the case $N_{n-1} < N_n$ we still use equation $(8.2.35_1)$ at the point $(N_n - 1, n)$. At the point (N_n, n) we use an equation arising from $(8.2.32_2)$ by replacement in its first part of τ by $\lambda\tau$, where $\lambda\tau$ is the distance between points (hN_n, t_n) and $(hN_n, T(N_n h))$. Here $t = T(x)$ is the equation of the polygon approximating the boundary on the interval (y_{n-1}, y_n). Finally, in the third case $(N_{n-1} > N_n)$ the equation for u_1 is approximated at the point $(N_n - 1, n)$ by the equation $(8.2.35_1)$ and at the point (N_n, n) by the equation $(8.2.32_2)$. In an analogous way we construct the approximation to the Fourier equation for u_2 at the points of the mesh adjoining the boundary.

The boundary conditions at the points $x = 0$ and $x = 1$ are approximated by the simplest equations

$$v_{1,\bar{x}}(0, t_m) = -q_{1,m}; \quad v_{2,\bar{x}}(1, t_m) = -q_{2,m} \tag{8.2.37}$$

for the condition that $h \leqq y_m \leqq 1 - h$. If $0 < y_m < h$, then the right side of (8.2.37) is replaced by the condition

$$v_1(0, t_m) = q_1(t_m)\, y_m. \tag{8.2.37*}$$

In an analogous way we can write the conditions at the point $x = 1$ for $1 - h < y_m$. In order to approximate the Stefan condition we introduce a parameter ϑ, $0 < \vartheta < \frac{1}{2}$. Let

$$y_n - N_n h \leqq \vartheta. \tag{8.2.38$_1$}$$

Then the Stefan condition is approximated by the expression

$$y_{\bar{t}}(t_n) = -\frac{v_1(N_n, n)}{y_n - (N_n - 1)h} - \frac{v_2(N_n + 1, n)}{(N_n + 1)h - y_n}. \tag{8.2.38$_2$}$$

If

$$(N_n + 1)h - y_n \leqq \vartheta, \tag{8.2.38$_3$}$$

then we take

$$y_{\bar{t}}(t_n) = -\frac{v_1(N_n, n)}{y_n - N_n h} - \frac{v_2(N_n + 2, n)}{(N_n + 2)h - y_n}. \tag{8.2.38$_4$}$$

It is obvious that the introduction of the parameter ϑ allows us to avoid difficulties at the points $y_n = N_n h$.

The system of equations approximating the Stefan condition is nonlinear. Hence the system of difference equations of the problem including the Stefan condition is nonlinear, and is solved by iteration.

The iteration scheme is completely analogous to that described for the second scheme.

The article [15] is of a preliminary character. As we have already noted, the authors announce a proof of the proposed scheme. However, the paper does not contain numerical results which might better characterize the method.

C. **Schemes with a fixed number of space nodes.** As we noted above, these schemes are investigated in two forms. The first form, described in the paper of M. Lotkin, and later in that of Budak and his collaborators, is based on the preliminary transformation of the domain of existence of each phase to an interval of fixed length. For this it is essential that the domain not be degenerate. For example, consider the one-phase problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}; \quad 0 < x < y(t); \quad t > 0;$$

$$u|_{x=0} = f(t); \quad u|_{t=0} = \phi(x); \quad u|_{x=y(t)} = 0; \tag{8.2.39}$$

$$\dot{y}(t) = -q(t) + \frac{\partial u}{\partial x}\bigg|_{x=y(t)}; \quad y(0) = 1.$$

The substitution

$$\xi = \frac{x}{y(t)}; \qquad u(\xi, t) = u(x, t) \tag{8.2.40}$$

changes it to the form

$$\frac{1}{y^2(t)}\frac{\partial^2 u}{\partial \xi^2} = \frac{\partial u}{\partial t} - \frac{\xi}{y(t)}\dot{y}(t)\frac{\partial u}{\partial \xi}; \quad 0 < \xi < 1; \quad t > 0;$$

$$u|_{\xi=0} = f(t); \quad u|_{t=0} = \phi(\xi); \quad u|_{\xi=1} = 0; \quad t > 0; \tag{8.2.41}$$

$$\dot{y}(t) = -q(t) + \frac{1}{y(t)}\frac{\partial u}{\partial \xi}\bigg|_{\xi=1}; \quad y(0) = 1.$$

Let $D_{h,\tau}$ be the mesh domain with lateral boundaries $\Gamma_{0,\tau}$ and $\Gamma_{1,\tau}$:

$$D_{h,\tau} = \{\xi_i = ih; \quad t_j = j\tau; \quad i = 1, 2, \cdots, N-1; \quad j = 1, 2, \cdots\};$$
$$\Gamma_{0,\tau} = \{\xi = 0; t_j\}; \quad \Gamma_{1,\tau} = \{\xi = 1; t_j\} \quad (Nh = 1). \tag{8.2.42}$$

Preserving the notation (8.2.4), we write the difference analog of the problem in the form

$$
\begin{aligned}
&\frac{1}{y^2} v_{\xi\bar{\xi}} = v_{\bar{t}} - \frac{\xi}{2y} y_{\bar{t}}(v_{\xi} + v_{\bar{\xi}}); \quad (\xi, t) \in D_{h,\tau};\\
&v = f, \quad (\xi, t) \in \Gamma_{0,\tau}; \quad v = 0, \quad (\xi, t) \in \Gamma_{1,\tau};\\
&v = \phi; \quad \xi = \xi_i; \quad t = 0;\\
&y_{\bar{t}} = -q(t) + \frac{1}{y} v_{\bar{\xi}}; \quad (\xi, t) \in \Gamma_{1,\tau}; \quad y(0) = 1.
\end{aligned}
\tag{8.2.43}
$$

The system (8.2.43) is nonlinear and is solved by iteration.

A very similar scheme is examined in the paper of Murray and Landis [**104**].

Let there be given an integer $N > 0$ and a smooth function $y(t)$. Set

$$
h(t) = \frac{y(t)}{N}\ ; \quad x_i(t) = ih(t) \quad (i = 0, 1, 2, \cdots, N). \tag{8.2.44}
$$

We have

$$
\frac{\partial}{\partial t} u(x_i(t), t) = \frac{d}{dt} u(x_i(t), t) - \dot{y}(t) \frac{\partial}{\partial x} u(x_i(t), t). \tag{8.2.45}
$$

We choose a mesh with nodal points

$$
\begin{aligned}
&(x_{i,j}, t_j); \quad x_{i,j} = x_i(t_j); \quad t_j = j\tau; \quad \tau = \text{const} > 0\\
&\qquad (i = 0, 1, 2, \cdots, N; \quad j = 0, 1, 2, \cdots).
\end{aligned}
\tag{8.2.46}
$$

Let $D_{N,\tau}$ be the mesh domain

$$
D_{N,\tau} = \{ x_{i,j}, t_j;\ i = 1, 2, 3, \cdots,\ N - 1; \quad j = 1, 2, 3, \cdots \} \tag{8.2.47}
$$

and $\Gamma_{0,\tau}$ and $\Gamma_{1,\tau}$ its lateral boundaries

$$
\Gamma_{0,\tau} = \{0; t_j\}; \quad \Gamma_{1,1,\tau} = \{ y(t_j); t_j \} \quad (j = 0, 1, 2 \cdots). \tag{8.2.47*}
$$

Then the system of difference equations approximating the equations of problem (8.2.39) can be written in the form

$$
\begin{aligned}
&u_{x\bar{x}} = u_{\bar{t}} - \frac{1}{2} y_{\bar{t}}(u_{\bar{x}} + u_x); \quad (x, t) \in D_{N,\tau};\\
&u = f, \quad (x, t) \in \Lambda_{0,\tau}; \quad u = 0, \quad (x, t) \in \Gamma_{1,\tau};\\
&u|_{t=0} = \phi; \quad x = x_{i,0}; \quad i = 0, 1, 2, \cdots, N;\\
&y_{\bar{t}} = -q + u_{\bar{x}}; \quad (x, t) \in \Lambda_{1,\tau}.
\end{aligned}
\tag{8.2.48}
$$

Here $u_x, \bar{u}_x$ and $\bar{u}_t$ are defined in agreement with (8.2.4) but with the replacement of the points (x_i, t_j) by the points $(x_{i,j}, t_j)$, τ_m by τ and h by $h_j = h(t_j)$.

D. **Schemes for the construction of the generalized solution of the Stefan problem.** We will examine the problem $(7.2.2_1)$-$(7.2.2_3)$, or, by virtue of the existence and uniqueness of the solution to problem (7.2.6), $(7.2.2_2)$, the equivalent problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial a(u)}{\partial t}; \quad 0 < x < 1; \quad u \neq s_i \quad (i = 1, 2, 3, \cdots, N);$$

$$u|_{x=0} = f_0(t); \quad u|_{x=1} = f_1(t); \quad u|_{t=0} = \phi(x); \tag{8.2.49}$$

$$a(u) = \int_{u^0}^{u} c(s)\, ds + \gamma_i \quad \text{for} \quad s_{i-1} < u < s_i; \quad \gamma_i > 0; \quad u^0 < s_1.$$

We define $A_0(x)$, for any point x where $\phi(x) \neq s_i$ $(i = 1, \cdots, N)$, by

$$A_0(x) = a(\phi(x)). \tag{8.2.50}$$

At every point x where $\phi(x) = s_i$ we set

$$A_0(x) = \frac{1}{2}\,[a(s_i - 0) + a(s_i + 0)]. \tag{8.2.51}$$

Furthermore, we set

$$a(u(x,t)) = A(x,t) \quad \text{for} \quad u \neq s_i. \tag{8.2.52}$$

Thus

$$A(x, 0) = A_0(x). \tag{8.2.53}$$

We now replace problem (8.2.49) by the finite-difference problem

$$u_{x\bar{x}} = A_t; \quad (x,t) \in D_{h,\tau'};$$

$$u = f_i; \quad (x,t) \in \Lambda_{i,\tau'} \quad (i = 0, 1); \tag{8.2.54}$$

$$A(x_i, 0) = A_0(x_i) \quad (i = 1, 2, \cdots, N-1).$$

Here $u_{x\bar{x}}$ and A_t are defined in agreement with (8.2.4), and τ' and h are positive constants. $D_{h,\tau}$ and $\Gamma_{i,\tau}$, as above, denote the mesh domain and its lateral boundaries:

$$D_{h,\tau} = \{x_i = ih; \ t_j = j\tau; \ i = 1, 2, \cdots, N-1; \ j = 1, 2, \cdots\},$$

$$\Gamma_{0,\tau} = \{x = 0; \ t = t_j\}; \quad \Gamma_{1,\tau} = \{x = 1; \ t = t_j; \ j = 0, 1, 2, \cdots\}. \tag{8.2.55}$$

The scheme (8.2.54) is explicit. Knowing $A(x_i, t_j)$ and using the monotonicity of $a(u)$, we determine $u(x_i, t_j)$. From the values $A(x_i, t_j)$ and $u(x_i, t_j)$ we find $A(x_i, t_{j+1})$, etc.

As we noted above, the scheme is highly accurate in general, and is certainly of value if

$$0 < \tau \leqq \mu h^2. \tag{8.2.56}$$

Here

$$\mu \leqq \frac{\eta(c_0 + 2)}{2}(1 - \beta); \quad c_0 = \max_{\substack{0 \leqq x \leqq 1; 0 \leqq t \leqq T; \\ i = 0,1}} \{ |\phi(x)|; \quad |f_i(t)| \};$$
$$0 < \eta \leqq \frac{a(u) - a(v)}{u - v} \tag{8.2.57}$$

and $\beta > 0$ is arbitrary.

The scheme is exceedingly simple but, like all explicit schemes, highly uneconomical. Thus for practical computation it is advantageous to use an implicit difference scheme with a priori smooth $a(u)$, corresponding to the presentation in Oleĭnik's theorem (Chapter VII, §2). Budak and his collaborators recommend the following method of smoothing. In the interval $s - \delta_1 < u < s + \delta_2$ about a point of discontinuity $u = s$ of the function $a(u)$ the function is replaced by a curve consisting of two parabolic arcs of fifth order lying between the endpoints of the interval. The parabolas are to be chosen so that at points of intersection we are guaranteed that the function has continuous first and second derivatives. Here it is recommended that the following requirements be satisfied:

a) The values of the temperature assumed on the intervals of smoothing correspond to 4—8 of the mesh nodes.

b) On each of the half-intervals $s - \delta_1 < u < s$ and $s < u < s + \delta_2$ of the values of the temperature roughly one-half of the indicated nodes must occur.

To attain these conditions a suitable variation of the smoothing parameters must be provided.

§3. Additional remarks on the method of integral equations [135]

The method of reduction of the Stefan problem to an integral equation with the use of a heat potential, developed in the earlier chapters, is first of all a theoretically oriented method of analyzing the problem.

As a method of numerical solution it is more complicated than reduction to ordinary differential equations or solution by finite difference methods. However, in a number of cases its application as a numerical method becomes necessary. Among such cases are those of the first boundary problem of Stefan with a degenerate initial domain of existence of one of the phases and discontinuity of boundary and initial conditions for a nonmonotonically varying boundary. In addition, when used as a numerical method of solution for the problem, the method of integral equations makes it possible in many cases to obtain a two-phase approximation with an a priori guaranteed degree of accuracy. The first aspect of the method was examined in Chapters III and V, in studying the initial behavior of the phase change process. Here we will briefly examine the second aspect.

For definiteness we will investigate problem B_1 (see Chapter II, §1):

$$
\begin{aligned}
&L_1 u = 0; \quad (x,t) \in D_1 = \{0 < x < y(t); \quad t > 0\};\\
&L_2 u = 0; \quad (x,t) \in D_2 = \{y(t) < x < 1; \quad t > 0\};\\
&u_1|_{x=0} = f_1(t); \quad u_i|_{t=0} = \phi(x); \quad u_2|_{x=1} = f_2(t); \qquad (8.3.1)\\
&u_i|_{x=y(t)} = 0;\\
&\dot{y}(t) = v_1(t) - v_2(t); \quad y(0) = l \in (0,1); \quad v_i = \frac{\partial}{\partial x} u_i(y(t),t).
\end{aligned}
$$

Assume that

$$
\begin{aligned}
&f_1(t) < 0; \quad f_2(t) > 0; \quad t > 0; \quad \phi_1(x) < 0; \quad 0 < x < l;\\
&\phi_2(x) > 0; \quad l < x < 1; \quad f_1(0) = \phi_1(0); \quad f_2(0) = \phi_2(1); \qquad (8.3.2)\\
&\phi_i(0) = 0 \quad (i = 1,2).
\end{aligned}
$$

Furthermore, define the operators

$$
\begin{aligned}
V_1(t|v_1, y) = &-2\int_0^t \dot{f}_1(\tau)\, g_2(y(t), 0, t-\tau)\, d\tau\\
&+2\int_0^l \dot{\phi}_1(\xi)\, g_2(y(t), \xi, t)\, d\xi \qquad (8.3.3_1)\\
&-2\int_0^t v_1(\tau)\frac{\partial}{\partial \xi} g_2(y(t), y(\tau), t-\tau)\, d\tau;
\end{aligned}
$$

$$V_2(t|v_2,y) = 2\int_0^t \dot{f}_2(\tau)\, G_2(y(t),1,t-\tau)\, d\tau$$

$$+ 2\int_l^1 \dot{\phi}_2(\xi)\, G_2(y(t),\xi,t)\, d\xi \tag{8.3.3$_2$}$$

$$+ 2a^2\int_0^t v_2(\tau)\frac{\partial}{\partial\xi} G_2(y(t),y(\tau),t-\tau)\, d\tau,$$

where g_2 and G_2 are defined in accordance with (0.2.6) and (0.2.9).

In agreement with Theorem 5 the solution v_1, v_2, y of problem B_1 is a solution of the system of integral equations

$$v_1 = V_1(t|v_1,y);\quad v_2 = V_2(t|v_2,y);\quad y = l + \int_0^t (v_1 - v_2)\, d\tau. \tag{8.3.3$_3$}$$

Furthermore, in agreement with Theorem 3, v_1, v_2, y will be given by the iteration process

$$v_{i,n+1}(t) = V_i(t|v_{i,n}, y_n) \quad (i = 1,2);$$
$$y_n(t) = l + \int_0^t (v_{1,n} - v_{2,n})\, d\tau, \tag{8.3.4}$$

where the choice of the zeroth approximations $v_{i,0}(t)$ and $y_0(t)$ $(y_0(0) = l)$ is not restricted. The sequence of approximations converges uniformly on a sufficiently small interval $(0, T)$; moreover, on this interval the sequence $\{\sqrt{t}\dot{v}_{i,n}(t)\}$ is uniformly bounded. We now prove

THEOREM 25. *Let f_i, ϕ satisfy the conditions* (8.3.2), *and in addition let*

$$\dot{f}_1 \leqq 0;\quad \dot{\phi}_i \geqq 0;\quad \ddot{\phi}_i \leqq 0 \quad (i = 1,2);\quad \dot{f}_2 \geqq 0. \tag{8.3.5}$$

Then the inequalities

$$v_{1,0} \geqq v_1;\quad v_{2,0} \leqq v_2;\quad \dot{y}_0 \geqq \dot{y};\quad y_0(0) = y(0) = l \tag{8.3.6}$$

imply the inequalities

$$v_{1,2n} \geqq v_1;\quad v_{2,2n} \leqq v_2;\quad \dot{y}_{2n} \geqq \dot{y};$$
$$v_{1,2n+1} \leqq v_1;\quad v_{2,2n+1} \geqq v_2;\quad \dot{y}_{2n+1} \leqq \dot{y} \tag{8.3.7}$$

for $0 \leqq t \leqq T$, where $T > 0$ is sufficiently small and $n \geqq 0$ is arbitrary.

In fact, from (8.3.5) and (0.2.6) it follows that

$$J_{1,1} = -2\int_0^t \dot{f}_1(\tau)\, G_1(y(t), 0, t-\tau)\, d\tau$$

$$= -\frac{2}{\sqrt{\pi}}\int_0^t \dot{f}(\tau)\, \frac{\exp(-y^2(t)/4(t-\tau))}{\sqrt{t-\tau}}\, d\tau \tag{8.3.8}$$

is a decreasing functional of y. Furthermore,

$$J_{1,2} = 2\int_0^l \dot{\phi}_1(\xi)\, G_1(y(t), \xi, t)\, d\xi$$

$$= \frac{2}{\sqrt{\pi}}\int_{(y-l)/2\sqrt{t}}^{y/2\sqrt{t}} \dot{\phi}_1(y + 2\zeta\sqrt{t}) \exp(-\zeta^2)\, d\zeta \tag{8.3.9}$$

$$+ \frac{2}{\sqrt{\pi}}\int_{y/2\sqrt{t}}^{(y+l)/2\sqrt{t}} \dot{\phi}_1(2\sqrt{t}\zeta - y) \exp(-\zeta^2)\, d\zeta.$$

Consequently

$$\frac{\partial J^2}{\partial y} = \frac{\dot{\phi}_1(l)}{\sqrt{\pi t}}\left[\exp\left(-\frac{(y+l)^2}{4t}\right) - \exp\left(-\frac{(y-l)^2}{4t}\right)\right]$$

$$+ \frac{2}{\sqrt{\pi}}\int_0^l \dot{\phi}_1(\xi)\left[\exp\left(-\frac{(y(t)-\xi)^2}{4t}\right)\right. \tag{8.3.10}$$

$$\left. - \exp\left(-\frac{(y(t)+\xi)^2}{4t}\right)\right]\frac{d\xi}{2\sqrt{t}}.$$

Hence from (8.3.5) it follows that $J_{1,2}$ is a decreasing functional of y.

Now let $t > 0$ be so small that for $0 \leqq t \leqq T$

$$0 \leqq \frac{y(t) - y(\tau)}{2\sqrt{t-\tau}} < \frac{1}{\sqrt{2}}; \qquad \frac{y(t) + y(\tau)}{2\sqrt{t-\tau}} > \frac{1}{\sqrt{2}}. \tag{8.3.11}$$

Since

$$\frac{\partial}{\partial\xi} g_2(y(t), y(\tau), t-\tau)$$

$$= \frac{1}{2\sqrt{\pi}(t-\tau)}\left\{\frac{y(t)-y(\tau)}{2\sqrt{t-\tau}} \exp\left[-\frac{(y(t)-y(\tau))^2}{4(t-\tau)}\right]\right. \tag{8.3.12}$$

$$\left. - \frac{y(t)+y(\tau)}{2\sqrt{t-\tau}} \exp\left[-\frac{(y(t)+y(\tau))^2}{4(t-\tau)}\right]\right\},$$

it follows that the functional

$$J_{1,3} = -2\int_0^t v_1(\tau)\frac{\partial}{\partial\xi}g_2(y(t),y(\tau),t-\tau)\,d\tau \tag{8.3.13}$$

is decreasing relative to v_1 and $\dot{y}$ on the set of nonnegative $v_1(t)$ and $y(t)\in(0,1)$.

Combining the results, we conclude that $V_1(t|v_1,y)$ is a decreasing functional of $v_1(t)$ and $\dot{y}(t)$ on the set of nonnegative v_1, and $y\in(0,l)$. Similarly, we conclude that under certain assumptions $V_2(t|v_2,y)$ is an increasing functional of $v_2(t)$ and $\dot{y}(t)$. Hence from the uniform boundedness of $\{\dot{y}_n\}$ it obviously follows that the theorem is proved. We note that for a choice of $v_{1,0},\dot{y}_0$ and $v_{2,0}$ majorizing (or minorizing) v_1, $\dot{y}$ and v_2, it is not required that we know the solution v_1,v_2,y. In fact, it is sufficient to find $v_{1,0},v_{2,0}$ and y_0 satisfying the conditions of Lemma 1 (Chapter II, §2).

Let $\delta>0$ be arbitrarily small, and let $M>0$ satisfy the inequalities

$$(l-\delta)M \geqq \max_{0\leqq x\leqq l}|\phi_1(x)|;\quad M-\delta\geqq\max_{0\leqq t\leqq T}|f_1(t)|. \tag{8.3.14}$$

We set

$$U^*_{1,\delta}=(M-\delta)\left(1-\frac{x}{z_\delta}\right);\quad U^*_{2,\delta}=0;\quad \dot{z}_\delta=\frac{M}{z_\delta};$$

$$z_\delta(0)=l+\delta; \tag{8.3.15}$$

$$U^*_i=\lim_{\delta\to0}U^*_{i,\delta};\quad z=\lim_{\delta\to0}z_\delta.$$

It is obvious that $U^*_{i,\delta}$ satisfies the conditions of Lemma 1 for arbitrary $\delta>0$. Consequently

$$y(t)\leqq z(t)\quad\text{for}\quad 0\leqq t\leqq T. \tag{8.3.16}$$

It is easily seen that in this case, for $T>0$ sufficiently small,

$$\dot{y}(t)\leqq\dot{z}(t)=\frac{M}{z(t)};\quad v_1(t)\leqq\frac{M}{z(t)};\quad v_2(t)\geqq0. \tag{8.3.17}$$

Thus we obtain a very simple but very rough majorizing estimate of the solution to the problem.

We note that, simultaneously with the Picard method for the solution of the integral equation form of the Stefan problem, we employ the method

of finite differences. Namely, we choose some $T > 0$ and divide the interval $(0, T)$ into a sequence of subintervals by means of the points $t_0 = 0 < t_1 < \cdots < t_n = T$.

We set

$$v_{i,n}(t_{k+1}) = V_i(t_k | v_{i,n}, y_n) \quad (i = 1, 2);$$

$$v_{i,n}(t) = v_{i,n}(t_k) + \frac{v_{i,n}(t_{k+1}) - v_{i,n}(t_k)}{t_{k+1} - t_k} (t - t_k) \text{ for } t_k \leqq t \leqq t_{k+1};$$

(8.3.18)

$$y_n(t) = l + \int_0^t (v_{1,n}(\tau) - v_{2,n}(\tau))\, d\tau.$$

The sequences $\{v_{i,n}(t)\}$ $(i = 1, 2)$ and $\{\dot{y}_n(t)\}$ are uniformly bounded and uniformly satisfy a Hölder condition with index $\frac{1}{2}$ if $0 \leqq t \leqq T$, where $T > 0$ is sufficiently small. Hence the convergence of the process to the solution of the problem follows for any subdivision of the interval into parts, provided that the length of each of the intervals of the subdivision tends to zero for $n \to \infty$.

In practice it is advisable to combine the finite difference and iteration processes. Namely, given a problem arising from concrete conditions, with the interval $(0, T)$ and the subdivision points t_k $(k = 0, \cdots, n, t_0 = 0, t_n = T)$, we determine the values $v^0_{i,n}(t_1)$, y_n by equation (8.3.18). We obtain values $v_{i,n}(t_1)$, y_1 and we subject them to as many iterations as may be necessary. After this we define $v^0_{i,n}(t_2)$, and perform the iteration process, etc. (see Chapter I, §6 A). It is obvious that this process can be improved. Thus linear interpolation can be replaced by parabolic. After the determination of $v_{i,n}(t_n)$ the values $v^0_{i,n}(t_{n+1})$ can be subjected to linear or parabolic extrapolation, attaining the maximum possible accuracy for a reduced number of iterations. Then the unknown solution lies between the approximation of two adjoining values, provided, of course, the condition which guarantees inequality (8.3.7) is valid.

For the choice t_k $(k = 0, \cdots, n)$ of the subdivision points it follows that $v_i(t) - v_i(0)$ is a function of the form $w_i(t)\sqrt{t}$, and w_i is bounded and, generally speaking, different from zero for $t = 0$. Therefore the length of the interval of the subdivision abutting on $t = 0$ should be minimal. But with increasing distance from $t = 0$, the length of the intervals of the subdivision can be extended.

For the numerical solution of the problem the integral is replaced by a finite sum, using any mechanical quadrature formula. The equation

obtained as a result of such a replacement, called below the "modification," is solved by the use of the finite difference process. Consequently the error in determining v_i arises from two sources: from the error due to replacement of the integral by a finite sum (approximation error), and from the error of halting the iteration for the solution to the modified system at some definite mesh (iteration error).[1] It is easily seen that as long as the conditions of Theorem 25 remain valid, inequality (8.3.7) will remain in effect for iterations of the modified equation. This shows that the error of iteration does not exceed the difference of the approximations of two neighboring values.

Generally speaking, the error of approximation can be estimated only indirectly, since we do not know any bounds on the highest derivatives of the function in a subinterval. Along with the usual applications of the device of estimating the accuracy by means of doubling the number of integral subdivisions, we can recommend comparison of the results of numerical integration with different formulas for mechanical quadrature. Thus, for example, if $w(t,\tau)/(t-\tau)^{1/2}$ is concave relative to τ, then the evaluation of the integral

$$J(t) = \int_{t_1}^{t_2} \frac{w(t,\tau)}{\sqrt{t-\tau}}\, d\tau \tag{8.3.19*}$$

by the trapezoidal rule leads automatically to an improved result. But if $J(t)$ is examined as a Stieltjes integral

$$J(t) = -2\int_{t_1}^{t_2} w(t,\tau)\, d\sqrt{t-\tau} \tag{8.3.19**}$$

and if $w(t,\tau)$ is convex relative to τ, then the trapezoidal formula

$$J(t) \cong \sum_{k=0}^{n-1} (w(t,\tau_k) + w(t,\tau_{k+1}))\,(\sqrt{t-\tau_k} - \sqrt{t-\tau_{k+1}}) \tag{8.3.19}$$

gives the result already stated. Thus when the listed conditions are realized, the error of approximation admits a definite, though rough estimate which can be used to choose a mesh length for a finite-difference iteration process for solving the integral equation form of the Stefan problem.

In making the choice of the quadrature formulas we note that calculation of integrals of the form

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[1] We do not speak here of the roundoff error. By virtue of the stability of the solution it is not of essential interest.

$$J_{1,i}^{t_1,t_2}(x,t,\xi,\tau|\,f(\tau)) = \int_{t_1}^{t_2} f(\tau)\,E(x+(-1)^i\xi,t-\tau)\,d\tau;$$

$$J_{2,i}^{t_1,t_2}(x,t,\xi,\tau|\,f(\tau)) = \int_{t_1}^{t_2} f(\tau)\,\frac{\partial}{\partial x}E(x+(-1)^i\xi,t-\tau)\,d\tau;$$

$$J_{3,i}(x,t,\xi,\tau,\alpha,\beta|\,f(\xi)) = \int_{\alpha}^{\beta} f(\xi)\,E(x+(-1)^i\xi,t)\,d\xi;$$

(8.3.20)

$$J_{4,i}^{t_1,t_2}(x,t,\xi,\tau,\alpha,\beta|\,f(\xi,\tau)) = \int_{t_1}^{t_2} d\tau \int_{\alpha}^{\beta} f(\xi,\tau)\,E(x+(-1)^i\xi,t-\tau)\,d\xi;$$

$$J_{5,i}^{t_1,t_2}(x,t,\xi,\tau,\alpha,\beta|\,f(\xi,\tau)) = \int_{t_1}^{t_2} d\tau \int_{\alpha}^{\beta} f(\xi,\tau)\,\frac{\partial}{\partial x}E(x+(-1)^i\xi,t-\tau)\,d\xi,$$

where $E(x,t)$ is defined in accordance with (0.2.4), must be carried out with care. Thus the application of the usual quadrature formulas can give a completely unsatisfactory result if the length of the subintervals of the subdivision is not sufficiently small. Practical computation shows that good results are invariably obtained by the following algorithm.

The integral $J_{1,i}^{t_1,t_2}$ is evaluated by formula (8.3.19). For the computation of $J_{2,i}^{t_1,t_2}$ we set

$$J_{2,i}^{t_1,t_2}(x,t,\xi,\tau|\,f(\tau)) = \sum_{k=0}^{n-1} \frac{f(\tau_k)+f(\tau_{k+1})}{2}\,J_{2,i}^{\tau_k,\tau_{k+1}}(x,t,\xi,\tau|\,1). \tag{8.3.21}$$

It is easily seen that

$$J_{2,i}^{\tau_k,\tau_{k+1}}(x,t,\xi,\tau|\,1) = \frac{1}{2}\left[\operatorname{erf}\frac{x+(-1)^i\xi(\tau_k)}{2\sqrt{t-\tau_k}} - \operatorname{erf}\frac{x+(-1)^i\xi(\tau_{k+1})}{2\sqrt{t-\tau_{k+1}}}\right] + (-1)^{i+1}J_{1,i}^{\tau_k,\tau_{k+1}}(x,t,\xi,\tau|\,\dot{\xi}(\tau)). \tag{8.3.22}$$

Assuming that the interval subdivision (τ_k,τ_{k+1}) is sufficiently small, we define $J_{1,i}^{k,k+1}$ by formula (8.3.19), setting $n=1$.

The integral $J_{3,i}$ is evaluated by the formula

$$J_{3,i}(x,t,\xi,\alpha,\beta|\,f(\xi)) = \sum_{k=0}^{n-1} \frac{f(\xi_k)+f(\xi_{k+1})}{2}\,J_{3,i}(x,t,\xi,\xi_k,\xi_{k+1}|\,1), \tag{8.3.23}$$

in which we take into account the fact that

$$J_{3,i}(x,t,\xi,\xi_k,\xi_{k+1}|\,1) = \frac{(-1)^i}{2}\left[\operatorname{erf}\frac{x+(-1)^i\xi_{k+1}}{2\sqrt{t}} - \operatorname{erf}\frac{x+(-1)^i\xi_k}{2\sqrt{t}}\right]. \tag{8.3.24}$$

For the evaluation of the integral $J_{4,i}^{t_1,t_2}$ the interior integral is determined by formula (8.3.23). Then the exterior integral is evaluated by the usual quadrature formulas, for example by the formula of Simpson or that of Gauss. Finally, the evaluation of $J_{5,i}^{t_1,t_2}$ is carried out by first changing the order of the integration, after which we evaluate the interior integral by formula (8.3.21), and the exterior by a general quadrature formula.

The use of this algorithm, in the sense that every integral is treated as a Cauchy-Stieltjes integral in which we use the trapezoidal rule, permits us to use a comparatively large interval for the subdivision; this not only compensates for complications of programming the calculation, but, in the final analysis, also means a saving of computing time. If we keep this in mind in our organization of the computations, it turns out that the method of integral equations is competitive with more direct methods for solving the Stefan problem, even in cases where these more direct methods may give satisfactory results. But this can be said if and only if the integral equation problem does not include double integrals of the form $J_{4,i}$ and $J_{5,i}$, since successive calculation of double integrals is especially difficult.

CHAPTER IX

PROBLEMS

We now formulate some still-unsolved problems which are of considerable interest.

A. **Crystallization of a binary (polycomponent) system.** In Part 1, Chapter II, §3, we investigated a problem, in similarity form, of crystallization of a binary system which is capable of forming a simple eutectic mixture. It was proved that for certain relations between the input data the problem does not have a similarity solution. It is required to investigate the problem in a nonsimilarity form. In particular, we are interested in the following problem.

At the initial moment suppose that both phases exist, while at the boundary between them the temperature and concentration are linked by the equations of a fluid. We pose the following problems: a) Whether in this case there exists a solution to the problem in the small? b) If the solution in the small does exist, under what conditions can it be extended to an arbitrary interval of time? c) If the solution cannot be extended to an arbitrary interval of time, then find the physical interpretation of this fact and describe the formulation of the problem which will guarantee the possibility of constructing the solution on any preassigned time interval. d) Answer the same question in the case of nonexistence of the solution to the original problem, not only in the large but also in the small.

In the same section we examined the problem of crystallization of a binary system generating a continuous solid slab. In this case G. P. Ivancov [66] discovered the effect called "diffusion supercooling." The validity of Ivancov's deductions are limited by the assumption that the coefficients of heat conduction and diffusion are constant, but in actual systems the coefficients of heat conduction are essentially dependent on the concentration, while the coefficients of diffusion are dependent on the temperature. In this connection it is of essential importance to develop the theory of crystallization of binary systems in a quasilinear treatment, taking this dependence into account. It is

essential that the theory be developed in a form admitting qualitative analysis of the problem.

B. **Unsteady flow of a viscous plastic medium.** We examine the boundary problem

$$L_1 u_1 = 0; \qquad 0 < x < y(t); \qquad t > 0; \tag{9.1$_1$}$$

$$L_2 u_2 = 0; \qquad y(t) < x < 1; \qquad t > 0; \tag{9.1$_2$}$$

$$u_1|_{x=0} = f_1(t); \quad u_i|_{t=0} = \phi(x); \quad u_2|_{x=1} = f_2(t) \qquad (i = 1, 2) \tag{9.1$_3$}$$

for the conditions on the unknown boundary

$$\frac{\partial u_i}{\partial x}\bigg|_{x=y} = \psi_i(y(t)) \qquad (i = 1, 2); \tag{9.2$_1$}$$

$$\frac{d}{dt} y = F\left(t, y, u_i, \frac{\partial u_i}{\partial x}, \frac{\partial u_i}{\partial t}\right)\bigg|_{x=y(t)} \qquad (i = 1, 2); \quad y(0) \in [0, 1] \tag{9.2$_2$}$$

or

$$u_i|_{x=y(t)} = \psi_i(y(t)); \tag{9.3$_1$}$$

$$y(t) = F\left(t, u_i; \frac{\partial u_i}{\partial x}\right)\bigg|_{x=y(t)}; \qquad y(0) \in [0, 1]. \tag{9.3$_2$}$$

Here ψ_i and F are given functions and L_1 and L_2 are one-dimensional parabolic operators.

As was shown in Part 1, Chapter IV, §3, problems of this type can arise in the theory of nonstationary flows of a viscous plastic medium. For a problem of this type it is necessary to derive a theory of existence, uniqueness and stability of the solution simultaneously with the definition of a numerical algorithm.

C. **Heat propagation in a medium with a concentration of capacity in which the phase change takes place.** The first of the problems formulated in B, above, is similar to the following general problem. Let there be given a distribution of temperature in a domain D bounded by the surface Σ consisting of two parts Σ_1 and Σ_2. Within D the temperature obeys the equation of heat conduction. On Σ_1 we are given the heat flux, the heat exchange with the surrounding medium, or the temperature. The surface Σ_2 is treated as a concentration of capacity [178], [142], along which heat propagates by means of conduction and liquid convection. Here we assume that the medium generating the concentration of capacity is two-phased, so that the surface Σ_2 is divided by a curve Γ which is subject to determination, on which the condi-

tions of continuity of temperature and of heat balance are similar to the ordinary Stefan conditions.

Problems of this type arise naturally in questions of thermal oil layers, when we examine the thermal injection in a stratum accompanying a phase transformation of the layer of liquid, or a chemical reaction in the solid matrix. An analogous problem arises naturally in the analysis of diffusion along the boundaries of cores accompanied by some kind of chemical reaction.

Apparently this type of problem has not been studied in the literature up to now. In the general statement of the problem formulated here, it is natural to pose the problem of defining the concept of generalized solution, and its existence and uniqueness. In the particular case of a plane problem, when the boundary Λ can be given by an equation of the form $x = x(t)$, the greatest interest lies in studying the existence of a classical solution. At the same time, it would be very interesting to develop a numerical algorithm for the solution of the problem.

D. **Analyticity of the interphase boundary.** Consider any of the problems $A_1, \cdots, C$, (see Part 2, Chapters II—IV). In these the boundary $x = y$ between phases depends on the time and the parameter $\alpha = a^{-1}$. In the thermal interpretation of the problem α can be considered as the ratio of the temperature conductivity of the solid and liquid phases.

In the known similarity solution of the Stefan problem the interphase boundary is a function of the form $\gamma t^{1/2}$. An example of an exact nonsimilarity solution of one particular Stefan problem was constructed by R. W. Sanders [154]; nevertheless in it the boundary is a function of the same form. Finally, in the solution of the problem of a flowing liquid the boundary varies linearly with time [184]. Thus in every known example of an exact solution of the Stefan problem the interphase boundary is analytic in $t^{1/2}$. Repeated attempts have been made to find the solution for more general problems and to prove the analyticity of the boundary as a function of $t^{1/2}$ for $t > 0$. None of these attempts has been successful. One is led to the problem: for what limitations on the boundary and initial conditions of the problem will the interphase boundary be analytic in $t^{1/2}$? The problem of determining such conditions is highly nontrivial. It is true that if an analytic curve $x = z(t)$ lies in the domain of the parabolic function $u(x,t)$, then in a neighborhood of it u is an analytic function of x and a function of the second Gevrey class with respect to t. Consequently $\partial u(z(t),t)/\partial x$ is not only

nonanalytic in the general case, but even non-quasi-analytic. This is even worse for $\partial u(z(t),t)/\partial x$ when $z(t)$ lies on the boundary of the domain of parabolicity of $u(x,t)$. Since $\dot{y}(t)$ is defined by the Stefan conditions in agreement with the value $\partial u(x,t)/\partial x$ for $x = y(t) \pm 0$, we see that generally speaking $y(t)$ cannot be analytic as a function of $t^{1/2}$.

A second, simpler problem is that of analyticity of y with respect to α. In the case of the problem with nondegenerate domain of existence of each phase the proof of the analyticity of the boundary with respect to the parameter α can be given by Gevrey's method of extending the solution to the complex plane [**50**] as used by A. Friedman for analyzing the solution of the problem of evaporation of a spherical drop [41]. Here it is almost obvious that all the estimates of Friedman are valid without change for the solution of problems B_1, B_3 and C. When we turn to problems A_1 and A_3, i.e. with a degenerate domain of initial existence of one of the phases, the problem is not solved so simply and requires a special examination.

The importance of carrying out a proof of the analyticity of the boundary with respect to the parameter is seen from the fact that in the work cited in the Introduction [173] Tadjakhsh and Lininger have shown that expanding the solution of the Stefan problem in a series of powers of a small parameter is in many cases a practicable method of constructing an approximate solution.

E. **Existence of a classical solution of the spatial Stefan problem.** The cardinal unsolved problem of the theory is the problem of the existence of a classical solution to the Stefan problem in the plane and spatial cases. The solution of these problems can be sought in two ways: a) by studying the structure of the set of all points of the space at which a generalized solution of the Stefan problem assumes the value $u = s_i$, appearing as points of discontinuity of the function $a(u)$ (see Chapter VII, §2), and b) by studying the problem directly, without reference to the theory of generalized solutions.

We note that, as was pointed out by O. A. Oleĭnik, who invented the method, the problem of the structure of the point set $u = s_i$ has not been studied at all. It seems to us that the extreme generality of the problem for which, with no greater complication than for the simplest one-dimensional Stefan problem, the existence of a generalized solution can be established, makes it very doubtful that this problem can be solved by the first method. In our opinion it is more probable that a direct proof can be given for the existence of a classical solu-

tion of the higher-dimensional Stefan problem, although perhaps only in its simplest form. We note that attempts to carry over the method of integral equations of the one-dimensional Stefan problem to the plane and spatial cases have been unsuccessful. In this connection we note that B. M. Budak, in one of his reports, announced a proof of convergence of the finite-difference algorithm "rectification of the front" (see Chapter VIII, §2C) to the solution of the Stefan problem in the spatial case, under the assumption of its existence.

SUPPLEMENT 1

THE MAXIMUM PRINCIPLE AND THE VÝBORNÝ-FRIEDMAN THEOREM

§1. Notation

In the following we will make use of the following notation:

$\mathscr{E}_n$ is n-dimensional Euclidean space.

$R_n = \mathscr{E}_n \times (-\infty < \tau < \infty)$ is the topological product of $\mathscr{E}_n$ with the line.

$(x_1, \cdots, x_n)$ are rectangular Cartesian coordinates in $\mathscr{E}_n$;

$(x_1, \cdots, x_n, \tau)$ are rectangular Cartesian coordinates in the space R_n; here the x-coordinates are space variables and τ is the time. The point $P = (x_1, \cdots, x_n, \tau)$ will be written as (p, τ) or (x, τ). A function of the point P will be written as $u(P)$ or $u(p, \tau)$ or $u(x, \tau)$.

Let D be a domain of R_n, lying between the planes $\tau = t_0$ and $\tau = t > t_0$, $\overline{D}$ the closure of D in R_n and $\overline{D}_{\tau_0}$ the intersection of $\overline{D}$ with the plane $\tau = \tau_0$. The set of interior points of $\overline{D}_{\tau_0}$ relative to $\mathscr{E}_n$ is denoted by D_{τ_0}. Assume that D_τ is a domain from $\mathscr{E}_n$ for $t_0 < \tau \leqq t$. We set

$$
\begin{aligned}
&D^t_{t_0} = \bigcup_{t_0 < \tau \leqq t} D_\tau; \quad \Sigma_\tau = \overline{D}_\tau \setminus D_\tau; \\
&\Sigma^t_{t_0} = \bigcup_{t_0 < \tau \leqq t} \Sigma_\tau; \quad \Gamma^t_{t_0} = \overline{\Sigma^t_{t_0} \cup D_{t_0}}.
\end{aligned} \tag{S1.1.1}
$$

We shall call $D^t_{t_0}$ a parabolic domain, $\Gamma^t_{t_0}$ its parabolic boundary, $\Sigma^t_{t_0}$ the lateral boundary and $D^t_{t_0}$ the base. For the examination of $D^t_{t_0}$ we will always consider t as a variable. In the case $t_0 = 0$ we will write D^t, Γ^t and Σ^t in place of $D^t_{t_0}$, $\Gamma^t_{t_0}$ and $\Sigma^t_{t_0}$, respectively.

Let Δ be the Laplace operator relative to the space coordinates, $a^2 = \text{const} > 0$ and $L_a \equiv a^2\Delta - \partial/\partial t$. If $u(p, \tau)$ has the derivatives in D^t which enter into L_a, we will say that U is a-superparabolic, a-parabolic or a-subparabolic in D^t, if

$$
L_a u \leqq 0, \quad L_a u = 0 \quad \text{or} \quad L_a u \geqq 0; \quad P \in D^t. \tag{S1.1.2}
$$

In the case $a^2 = 1$ we will refer simply to a superparabolic, parabolic

or subparabolic function. Along with the operator L_a we will consider the operator

$$L^*u = \operatorname{div}[k(p,\tau)\operatorname{grad} u] + \sum_{i=1}^{n} b_i(p,\tau)\frac{\partial u}{\partial x_i} - c(p,\tau)\frac{\partial u}{\partial \tau}, \tag{S1.1.3}$$

assuming c and b_i $(i = 1, 2, 3, \cdots, n)$ are continuous and k continuously differentiable in D^t with respect to the spatial coordinates, where $c \geqq c_0 > 0$ and $k \geqq k_0 > 0$.

§2. The weak maximum principle

THEOREM S1.1. *Let D^t be a bounded parabolic domain, let u be continuous in $\overline{D}^t$ and $L^*u \geqq 0$ $(L^*u \leqq 0)$ within D^t. Then the maximum (minimum) of u is attained on Γ^t.*

Let us assume the contrary, i.e. that

$$\begin{aligned} &L^*u \geqq 0; \quad (p,\tau) \in D^t; \\ &m = \max_{(p,\tau)\in\Gamma^t} u < M = \max_{(p,\tau)\in\overline{D}^t} u. \end{aligned} \tag{S1.2.1}$$

Then the value M is assumed at some point $(p_0, \tau_0) \in D^t$. We examine the function

$$v(p,\tau) = u(p,\tau) - \frac{M-m}{2\tau_0}(\tau - \tau_0) \tag{S1.2.2}$$

in D^{τ_0}. We have

$$\begin{aligned} &v \leqq m + \frac{M-m}{2} < M \text{ for } (p,\tau) \in \Gamma^{\tau_0}; \\ &v(p_0,\tau_0) = M. \end{aligned} \tag{S1.2.3}$$

Consequently the maximum of v is attained at some point $(q,\tau) \in D^{\tau_0}$. At this point $\operatorname{grad} v = 0$, $\Delta v \leqq 0$ and $\partial v/\partial t \geqq 0$; hence

$$L^*[v(q,\tau)] \leqq 0, \tag{S1.2.4}$$

which is impossible, since by the definition of v it follows that everywhere in D^{τ_0}

$$L^*v = L^*u + \frac{M-m}{2\tau_0} > 0. \tag{S1.2.5}$$

The theorem is proved.

REMARK 1. Let u be continuous in $\overline{D}^t$ and within D^t let it satisfy the condition $L^*u + \delta u \leqq 0$ $(L^*u + \delta u \geqq 0)$ with $\delta \leqq 0$. Then the negative

minimum (positive maximum) of u is attained on Γ^t. (The proof is obvious.)

REMARK 2. If $k \equiv a^2 = \text{const} > 0$, $b_i \equiv 0$ $(i = 1, \cdots, n)$ and $c \equiv 1$, then the operator L^* coincides with the operator L_a. Consequently the maximum (minimum) of a continuous a-subparabolic (a-superparabolic) function within D^t is attained on Γ^t.

THEOREM S1.2. *Let D^t be a bounded parabolic domain such that Σ_0 can be contained within a domain of arbitrarily small n-dimensional measure. Further, let u be a uniformly bounded function in D^t, continuous in $\overline{D}^t \setminus \Sigma_0$ and satisfying within D^t the condition $L^*u \geqq 0$ $(L^*u \leqq 0)$. Then the value u within D^t does not exceed (is not less than) the least upper (greatest lower) bound of its value on Γ^t.*

We limit ourselves to proving the theorem for subparabolic functions. The proof is given for a general linear parabolic operator, and hence in particular for the operator L^*, introduced for example in [62].

Thus for u continuous in $\overline{D}^t \setminus \Sigma_0$ we have

$$\text{(S1.2.6)} \qquad \sup_{(p,\tau)\in\Gamma^t\setminus\Sigma_0} u = m; \quad \sup_{(p,\tau)\in D^t} |u| = M > 0; \quad Lu \geqq 0; \quad (p,\tau) \in D^t.$$

Without loss of generality, we may assume that $m = 0$. Otherwise we examine $v = u - m$. We will show that if (p_0, t) is any point of D_t, then $u(p,t) \leqq 0$. In fact, from the continuity of u in $\overline{D}^t \setminus \Sigma_0$ it follows that for any $\epsilon > 0$ we can find some $\eta_0 < t$ such that for any fixed $\eta \in (0, \eta_0)$ and domain $G_\eta \subset \mathscr{E}_n$ such that $\overline{G}_\eta \subset D_\eta$ we will have

$$\text{(S1.2.7)} \qquad u(p,\tau) < \epsilon; \quad (p,\tau) \in \overline{G}_\eta \times (0 \leqq \tau \leqq \eta_0).$$

We now choose a domain $G_{i,\eta}$ $(i = 1, 2, 3)$ in $\mathscr{E}_n$ determined by the conditions

$$\text{(S1.2.8)} \qquad \overline{G}_{1,\eta} \subset G_{2,\eta} \subset \overline{G}_{2,\eta} \subset D_\eta \subset \overline{D}_\eta \subset G_{3,\eta};$$

$$\text{(S1.2.9)} \qquad \begin{cases} \operatorname{mes} G_{3,\eta} \setminus D_\eta = \mu_3 < \dfrac{\epsilon}{3M} \left(2a\sqrt{\pi(t-\eta)}\right)^n = \lambda\epsilon; \\ \operatorname{mes} D_\eta \setminus G_{2,\eta} = \mu_2 < \lambda\epsilon; \\ \operatorname{mes} G_{2,\eta} \setminus G_{1,\eta} = \mu_1 < \lambda\epsilon. \end{cases}$$

This is possible, since by the assumptions D^t is bounded and Σ_0 can be included within a domain of arbitrarily small n-dimensional measure.

Furthermore, we define a function $\psi_n(p)$, continuous in $\mathscr{E}_n$, by the conditions

$$\psi_\eta(p) \equiv 0; \quad p \in \mathscr{S}_n \setminus G_{3,\eta};$$
$$\psi_\eta(p) \equiv M; \quad p \in D_\eta \setminus G_{2,\eta};$$
(S1.2.10) $$\psi_\eta(p) \equiv \epsilon; \quad p \in G_{1,\eta};$$
$$0 < \psi_\eta(p) < M; \quad p \in G_{3,\eta} \setminus D_\eta;$$
$$\epsilon < \psi_\eta(p) < M; \quad p \in G_{2,\eta} \setminus G_{1,\eta};$$

We set

(S1.2.11) $$\Pi_\eta(p,\tau) = \int_{\mathscr{S}_n} \psi_\eta(q)\,(2\sqrt{\pi(t-\eta)})^{-n} \exp\left(-\frac{r_{pq}^2}{4(t-\eta)}\right) d\omega_q.$$

$\Pi_\eta(p,\tau)$ is defined in $\overline{D}_\eta^t$, and moreover

$$\Pi_\eta(p,\tau) \geqq 0; \quad (p,\tau) \in \Sigma_\eta^\tau;$$
(S1.2.12) $$\Pi_\eta(p,\tau) \geqq \epsilon; \quad (p,\tau) \in D_\eta;$$
$$L[\Pi_\eta(p,\tau)] = 0; \quad (p,\tau) \in D_\eta^\tau.$$

Comparing (S1.2.12) and (S1.2.6), we find that if

(S1.2.13) $$v(p,\tau) = u(p,\tau) - \Pi_\eta(p,\tau),$$

then

(S1.2.14) $$Lv \geqq 0; \quad (p,\tau) \in D_\eta^t; \quad v \leqq 0; \quad (p,\tau) \in \Gamma_\eta^t,$$

and moreover $v(p,\tau)$ is continuous in $\overline{D}_\eta^t$. Hence from Theorem S1.1 it follows that $v(p,\tau) \leqq 0$ for $(p,\tau) \in D_\eta^t$, i.e. that

(S1.2.15) $$u(p,\tau) \leqq \Pi_\eta(p,\tau); \quad (p,\tau) \in D_\eta^t.$$

But from (S1.2.9)—(S1.2.11) it follows that

(S1.2.16) $$\Pi_\eta(p_0,t) < M(2a\sqrt{\pi(t-\eta)})^{-n}(\mu_1+\mu_2+\mu_3) + \epsilon \int_{G_{1,\eta}} (2a\sqrt{\pi(t-\eta)})^{-n} \exp\left(-\frac{r_{p_0q}^2}{4a(t-\eta)}\right) d\omega_q < 2\epsilon.$$

Inserting (S1.2.16) into (S1.2.15), we obtain

(S1.2.17) $$u(p_0,t) \leqq 2\epsilon.$$

Since $\epsilon > 0$ is arbitrarily small and (p_0,t) is an arbitrary point of D_t, we find that

(S1.2.18) $$u(p,\tau) \leqq 0; \quad (p,\tau) \in D^t,$$

which completes the proof.

REMARK. Theorem S1.2 remains valid for an unbounded domain D^t, if to the conditions of the theorem we add the condition that $U(p,t)$ tends uniformly to 0 at ∞. In fact, in this case it suffices to examine the cylinder

$$Q \equiv \left\{ \sum_{i=1}^{n} x_i^2 \leqq R^2; \quad 0 < \tau < \infty \right\} \tag{S1.2.19}$$

and its intersection

$$D^{*t} = Q \cap D^t \tag{S1.2.19*}$$

with the domain D^t. Let Γ^{*t} be the parabolic boundary of D^{*t} and $S = \Gamma^{*t} \setminus \Gamma^t$. Then for any $\epsilon > 0$ we can find $R > 0$ so large that

$$|u(p,\tau)| < \epsilon \text{ on } S. \tag{S1.2.20}$$

Repeating the proof of the theorem for the domain D^{*t} and taking (S1.2.20) into account, we establish that for any point $(p,\tau) \in D^{*t}$

$$u(p,\tau) \leqq \epsilon. \tag{S1.2.21}$$

Since for any point $(p,\tau) \in D^t$ we can find some R so large that $(p,\tau) \in D^{*t}$, and since $\epsilon > 0$ is arbitrarily small, we see that the theorem is in fact valid for unbounded domains.[1)]

§3. The strong maximum principle of Nirenberg [106]

THEOREM S1.3. *Let u be continuous in the closure of the bounded parabolic domain D^t and a-subparabolic (a-superparabolic) in D^t, and let it attain its maximum (minimum) M in an interior point $(p_0,\tau_0) \in D^t$. Then $u(p,\tau) \equiv M$ on the set $C(p_0,\tau_0)$ of all points (p,τ) from D^t which can be joined to (p_0,τ_0) by a continuous curve L: $\{x_i = x_i(\tau);\ i = 1,\cdots,n\}$, belonging to D^t, along which τ is not increasing as (p,τ) approaches (p_0,τ_0).*

REMARK. The theorem is valid for a general linear parabolic operator, and in particular for the operator L^*. The proof given below is due to Nirenberg. For a second proof see [62].

LEMMA S1.1. *Let $L_a u \geqq 0$ in D^t and let $\max_{D^t} u = M$ be attained at the point (p_0,τ_0) of the boundary S of a sphere K belonging, together with S, to D^t; moreover, within K let $u(p,\tau) < M$. Then the x-coordinates of the point (p_0,τ_0) coincide with the x-coordinates of the center of the sphere K.*

[1)] The proofs of Theorems S1.1 and S1.2 are due to A. N. Tihonov [177].

In fact, let us place the origin at the center of K. Without loss of generality we may assume that (p_0, τ_0) is the unique point of S at which u admits the value M.

For an arbitrary point $(p, \tau) \equiv (x_1, \cdots, x_n, \tau)$ we will write

$$\rho^2 = \sum_{i=1}^{n} x_i^2; \quad r^2 = \rho^2 + \tau^2. \tag{S1.3.1}$$

Thus r is the distance of (p, τ) from the center of K.

We will assume, in contradiction to the assertion of the lemma, that the x-coordinates of (p_0, τ_0) do not coincide with the x-coordinates of the center of K, i.e. that $\rho_0^2 > 0$. In this case we automatically have $\tau_0 < t$, or $\overline{K} \subset D^t$. This shows that there exists an $r_1 > 0$ so small that the sphere K_1 with center at (p_0, τ_0) and radius $r_1 < \rho_0$ belongs, together with its boundary S_1, to D^t.

The sphere K divides S_1 into two parts S_1' and S_1'', of which the first belongs to $\overline{K}$ and the second to the complement of $\overline{K}$. Since (p_0, τ_0) is the unique point of $\overline{K}$ at which $u = M$, there exists some $\eta > 0$ such that

$$u(p, \tau) \leqq M - \eta; \quad (p, \tau) \in S_1'. \tag{S1.3.2}$$

At the same time

$$u(p, \tau) \leqq M; \quad (p, \tau) \in S_1''. \tag{S1.3.3}$$

Let us set

$$h(p, \tau) = \exp\{-\alpha(\rho^2 + \tau^2)\} - \exp\{-\alpha r_0^2\}, \tag{S1.3.4}$$

where r_0 is the radius of K and $\alpha = \text{const} > 0$ will be chosen later. We have

$$\begin{aligned} &h(p, \tau) > 0; \quad (p, \tau) \in K; \\ &h(p, \tau) = 0; \quad (p, \tau) \in S; \\ &h(p, \tau) < 0; \quad (p, \tau) \in D^t \setminus \overline{K}. \end{aligned} \tag{S1.3.5}$$

In addition

$$L_a h = 2 \exp[-\alpha(\rho^2 + \tau^2)] \cdot \{2a^2\alpha^2\rho^2 - na^2\alpha + \alpha\tau\}. \tag{S1.3.6}$$

By our assumptions $r_1 < \rho_0$. Consequently $\rho > 0$ for the point $(p, \tau) \in K_1$. Therefore it is possible to take $\alpha > 0$ so large that

$$L_a h > 0; \quad (p, \tau) \in \overline{K}_1. \tag{S1.3.7}$$

We set

(S1.3.8) $$v(p,\tau) = u(p,\tau) + \epsilon h(p,\tau); \quad \epsilon = \text{const} > 0.$$

By virtue of (S1.3.7) and the conditions of the lemma

(S1.3.9) $$L_a v = L_a u + \epsilon L_a h > 0.$$

Consequently, by virtue of the weak maximum principle, v attains its maximum on S_1. From (S1.3.5) it follows that

(S1.3.10) $$v \leqq M + \epsilon h < M \text{ for } (p,\tau) \in S_1''.$$

Furthermore, on S_1'

(S1.3.10*) $$v \leqq M - \eta + \epsilon h.$$

Consequently it is possible to take $\epsilon > 0$ so small that

(S1.3.11) $$v < M \text{ for } (p,\tau) \in S_1'.$$

For such a choice of $\epsilon > 0$ we have $v < M$ throughout S_1. But this is impossible, since $v = M$ at the center (p_0, τ_0) of the sphere K_1. The lemma is proved.

LEMMA S1.2. *Let u be a-subparabolic in D^t and attain its maximum M at the point $(p_0, \tau_0) \in D^t$; moreover, let $\tau_0 < t$. Then*

(S1.3.12) $$u(p,\tau) = \text{const} = M(p,\tau) \in D_{\tau_0}.$$

In fact, let $\mathscr{E}$ be the set of all those points of D_{τ_0} at which $u(p,\tau) = M$. $\mathscr{E}$ is closed relative to D_{τ_0}. We suppose, in contradiction to the assertion of the lemma, that $D_{\tau_0} \setminus \mathscr{E}$ is not empty; then it contains a point (q_0, τ_0) whose distance ρ_0 to the set $\mathscr{E}$ must be less than its distance to the boundary Σ_{τ_0} of the domain D_{τ_0}. Thus $u(q_0, \tau_0) < M$, but by continuity there is a neighborhood U of the point (q_0, τ_0) contained in D^t and such that

(S1.3.13) $$u(p,\tau) < M \text{ for } (p,\tau) \in U.$$

We choose an interval Γ of the line, parallel to the τ-axis, with midpoint at the point (q_0, τ_0), such that

(S1.3.14) $$\Gamma \subset U,$$

and on Γ, as axis of rotation, we construct the set $\{A_\rho\}$ of ellipses A_ρ with boundaries $S_\rho = \overline{A}_\rho > A_\rho$ and with x-semiaxes of lengths ρ varying from 0 to ρ_0. Since $A_0 = \Gamma$, we have $u < M$ in $\overline{A}_0$. By the definition of ρ_0 there is a point on S_{ρ_0} at which $u = M$. Thus there is a $\lambda \in [0, \rho_0]$ such that

(S1.3.15) $$u < M \text{ for } (p, \tau) \in A_\lambda,$$

and moreover, on S_λ there is a point (q_1, τ_1) such that

(S1.3.16) $$u(q_1, \tau_1) = M.$$

We now inscribe in A_λ a sphere K touching S_λ at a point (q_1, τ_1). We then can apply Lemma S1.1. Thus the x-coordinates of the center of K coincide with the x-coordinates of the point (q_1, τ_1), which is possible only when (q_1, τ_1) is the vertex of the ellipse A_λ, coincident with one of the endpoints of the line Γ. Thus

(S1.3.17) $$u(q_1, \tau_1) = M; \quad (q_1, \tau_1) \in \Gamma \subset U.$$

But this contradicts (S1.3.13). The lemma is thus proved.

PROOF OF THEOREM S1.2. Let

(S1.3.18) $$L_a u > 0 \text{ in } D^t; \quad \max_{(p,\tau)\in D^t} u = u(p_0, t) = M \text{ and } (q_0, \tau_0) \in C(p_0, t).$$

We examine an admissible curve $L \subset D^t$ connecting the points (q_0, τ_0) and (p_0, t). We assume, in contradiction to the assertion of the theorem, that

(S1.3.19) $$u(q_0, \tau_0) < M; \quad \tau_0 < t.$$

Then on L there is a point (q_1, τ_1) nearest to (q_0, τ_0) at which $u = M$. Since $\tau_0 < \tau_1 < t$ and D_{τ_1} is a domain in $\mathscr{E}_n$, it is possible to construct two n-dimensional cubes Q_1 and Q_2 with centers at the point (q_1, τ_1), belonging to the plane $\tau = \tau_1$, and on them, as bases, two $n+1$ dimensional parallelograms R_1 and R_2 such that

(S1.3.20) $$\overline{Q}_2 \subset Q_1; \quad \overline{R}_2 \subset \overline{R}_1 \subset D^{\tau_1},$$

with moreover the point $(q_0, \tau_0) \in D^{\tau_1} \setminus \overline{R}_1$.

By virtue of Lemma S1.2

(S1.3.21) $$u(p, \tau) < M; \quad (p, \tau) \in R_1,$$

since in the opposite case there would be a point on L which is nearer to (q_0, τ_0) than (q_1, τ_1), and at which u is equal to the value M; but this is impossible, by construction.

We now examine the sphere K with radius r and boundary S, touching D_{τ_1} at the point (q_1, τ_1) and containing the point (q_0, τ_0). Let $(x_1^*, \cdots, x_n^*, \tau^*)$ be the coordinates of the center of K. We examine the function

(S1.3.22) $$h(p, \tau) = r^2 - \sum_{i=1}^{n} (x_i - x_i^*)^2 - (\tau - \tau^*)^2.$$

We have

$$L_a h = 2\{\tau - \tau^* - 2a^2 n\}. \tag{S1.3.23}$$

In $\overline{R}_1$ we have $\tau > \tau^*$. Thus it is possible to choose r so large that

$$L_a h > 0 \text{ for } (p,\tau) \in \overline{R}_1. \tag{S1.3.24}$$

We set

$$v(p,\tau) = u(p,\tau) + \epsilon h(p,\tau), \tag{S1.3.25}$$

where $\epsilon = \text{const} > 0$ will be chosen later.

Let Γ be a portion of the boundary of R_2 belonging to K, and C a part of K belonging to R_2. Since every point of Γ is an interior point of R_1 such that $u < M$ on Γ, and since the distance from Γ to (q_1,τ_1) is strictly positive, there is an $\eta > 0$ such that

$$u(p,\tau) \leqq M - \eta \text{ for } (p,\tau) \in \overline{\Gamma}. \tag{S1.3.26}$$

Hence for $\epsilon > 0$ sufficiently small

$$v(p,\tau) < M \text{ for } (p,\tau) \in \overline{\Gamma}. \tag{S1.3.27}$$

Furthermore, $h = 0$ on C; consequently

$$\begin{aligned} &v < M \text{ for } (p,\tau) \in C; \quad (p,\tau) \neq (q_1,\tau_1); \\ &v(q_1,\tau_1) = M. \end{aligned} \tag{S1.3.28}$$

Finally, throughout the domain $G = K \cap R_2$

$$L_a v = L_a u + \epsilon L_a h > 0. \tag{S1.3.29}$$

From (S1.3.27)—(S1.3.29) and the weak maximum principle, it follows that

$$v(p,\tau) \leqq M; \quad (q,\tau) \in \overline{G}. \tag{S1.3.30}$$

But this shows that

$$\frac{\partial}{\partial \tau} v(q_1,\tau_1) \geqq 0. \tag{S1.3.31}$$

Therefore we have

$$\frac{\partial}{\partial \tau} v(q_1,\tau_1) = \frac{\partial}{\partial \tau} u(q_1,\tau_1) - 2\epsilon(\tau_1 - \tau^*). \tag{S1.3.32}$$

But by construction $\tau_1 - \tau_1^* > 0$. Thus

(S1.3.33) $$\frac{\partial}{\partial \tau} u(q_1, \tau_1) \geqq 2\epsilon(\tau_1 - \tau^*) > 0.$$

Furthermore, at the point (q_1, τ_1), as at the maximum point of $u(p, \tau)$, we have

(S1.3.34) $$\Delta u \leqq 0.$$

From (S1.3.33) and (S1.3.34) it follows that

(S1.3.35) $$L_a u(q_1, \tau_1) = a^2 \Delta u - \frac{\partial u}{\partial \tau} < 0,$$

which contradicts the assumptions. The theorem is proved.

§4. Theorem of Výborný-Friedman [44], [196]

Below we formulate and prove a theorem due to A. Friedman. An analogous theorem was proved in [62].

THEOREM S1.3. *Let u be continuous in the closure of the parabolic domain D^t and a-subparabolic (a-superparabolic) within it. Let $(q_0, \tau_0) \in \Gamma^t$ be a point at which u attains its maximum (minimum) M. Assume that there exists a sphere K with the following properties:*

α_1: $$(q_0, \tau_0) \in \overline{K} \setminus K = S;$$

α_2: $$\overline{K^t} \subset \overline{D^t};$$

α_3: *the radius drawn to the point (q_0, τ_0) is not parallel to the τ axis;*

α_4: $$u(p, \tau) < M \quad (u > M) \text{ for } (p, \tau) \in K^t.$$

Let $\vec{\nu}$ be a direction issuing from a point (q_0, τ_0) within K. Assume that $\partial u(q_0, \tau_0)/\partial \nu$ exists. Then

(S1.4.1) $$\frac{\partial}{\partial \nu} u(q_0, \tau_0) < 0 \quad \left(\frac{\partial}{\partial \nu} u(q_0, \tau_0) > 0 \right).$$

REMARK. The theorem is valid for a general linear parabolic operator. The proof given here is a reproduction of Friedman's original proof, with appropriate simplifications.

PROOF. We transfer the origin to the center of K, and construct a cylinder C whose axis coincides with the time axis, and whose radius $\delta > 0$ is so small that

(S1.4.2) $$(q_0, \tau_0) \in \overline{G} = \overline{K} \setminus C.$$

Let

(S1.4.3) $$S_g = \overline{G} \setminus G; \quad S^t = \overline{K}^t \setminus K^t$$

be the boundaries of G and K^t respectively. We set

(S1.4.4) $$S_g' = S_g \cap S; \quad S_g'' = S_g \cap \overline{C}.$$

We examine the functions

(S1.4.5) $$h = \exp\{-\alpha(\rho^2 + \tau^2)\} - \exp(-\alpha R^2) \equiv h^* - \exp(-\alpha R^2),$$

where R is the radius of K and $\rho^2 = \sum_{i=1}^n x_i^2$. We have

(S1.4.6) $$L_a h = 2\alpha a^2 h^* \left(2\alpha\rho^2 - n + \frac{1}{a^2}\tau\right).$$

Since $\rho \geqq \delta > 0$ for $(p, \tau) \in G$, we see that for $\alpha > 0$ sufficiently large

(S1.4.7) $$L_a h > 0; \quad (p, \tau) \in G.$$

In addition

(S1.4.8) $$h > 0; \quad (p, \tau) \in K.$$

We set

(S1.4.9) $$v(p, \tau) = u(p, \tau) + \epsilon h(p, \tau).$$

We then have

(S1.4.10) $$L_a v = L_a u + \epsilon L_a h > 0; \quad (p, \tau) \in G,$$

since by assumption $L_a u \geqq 0$. Furthermore,

(S1.4.11) $$v(p, \tau) = u(p, \tau) \leqq M \text{ for } (p, \tau) \in S_g'.$$

Finally, by virtue of the facts that $u < M$ for $(p, \tau) \in S_g''$, and that the distance from S_g'' to (q_0, τ_0) is not less than $\delta > 0$, we see that there exists some $\eta > 0$ such that

(S1.4.12) $$u(p, \tau) \leqq M - \eta \text{ for } (p, \tau) \in S_g''.$$

Consequently for $\epsilon > 0$ sufficiently small

(S1.4.13) $$v(p, \tau) \leqq M - \eta + \epsilon h < M \text{ for } (p, \tau) \in S_g''.$$

By virtue of (S1.4.10)—(S1.4.13) and the strong maximum principle we have

(S1.4.14) $$v(p,\tau) < M; \quad (p,\tau) \in G.$$

This shows that

(S1.4.15) $$\frac{\partial}{\partial \nu} v(q_0,\tau_0) \leqq 0,$$

since $\vec{\nu}$ is directed into G and $v(q_0,\tau_0) = u(q_0,\tau_0) = M$. But from (S1.4.15) and (S1.4.9) it follows that

(S1.4.16) $$\frac{\partial}{\partial \nu} u(q_0,\tau_0) \leqq -\epsilon \frac{\partial}{\partial \nu} h(q_0,\tau_0).$$

We have

(S1.4.17) $$\frac{\partial}{\partial \nu} h(q_0,\tau_0) = -\alpha R \exp(-\alpha R^2)\cos(\vec{R},\vec{\nu}),$$

where $\vec{R}$ is a radius vector to the point (q_0,τ_0) with initial point at the center of K. Since $\vec{\nu}$ is directed from (q_0,τ_0) into K, we find that $\cos(\vec{R},\vec{\nu}) < 0$. This shows that

(S1.4.18) $$\frac{\partial}{\partial \nu} u(q_0,\tau_0) < 0,$$

which completes the proof.

REMARK. If the point of the boundary maximum (minimum) belongs to D_0, then the inequality (S1.4.18) cannot be valid, since at such a point condition α_3 automatically fails to hold. The essential importance of this fact is evident from the following example. Let $u = -\tau^2$. Then $L_a u = 2\tau > 0$ for $\tau > 0$ and the boundary maximum of u, equal to zero, is attained on D_0. At the same time, $D^t u < 0$ and $\partial u/\partial \tau|_{D_0} = 0$.

SUPPLEMENT II

THE LIMITING BOUNDARY VALUES OF THE DERIVATIVE OF A PARABOLIC FUNCTION. GREEN'S FUNCTION OF A DOUBLE-LAYER BOUNDARY PROBLEM ON THE LINE AND HALFLINE

§1. The limiting boundary values of the derivative of a parabolic function

In our examination of problems A_i, B_i and C in Chapters II–IV of Part 2, and in our investigation of the double layer single-phase Stefan problem in Chapter V of Part 2, the values of $\partial u(x,t)/\partial x$ at points of the boundary $x = y(t)$ were always regarded as the limiting values on approach to the boundary from within the domain of definition of $u(x,t)$. At the same time, in some places of the text it was estimated as the actual boundary values without proof of the possibility of such an estimate (see for example the proof of Lemma 1 in Part 2, Chapter II).

The legitimacy of this procedure follows from the almost obvious Theorem S2.1:

THEOREM S2.1. *Let $u(x,t)$ be continuous in the closure of the parabolic domain*

$$D^t = \{X_1(\tau) < x < X_2(\tau); \quad 0 < \tau \leqq t\} \tag{S2.1.1}$$

everywhere with the possible exception of the points $(X_i(0), 0)$ $(i = 1,2)$, and bounded and parabolic in D^t, and let it satisfy the conditions

$$u|_{x=X_i(t)} = f_i(t) \quad (i = 1,2); \quad u|_{t=0} = \phi(x); \quad X_1(0) < x < X_2(0), \tag{S2.1.2}$$

where X_i, f_i and ϕ are differentiable for $0 \leqq \tau \leqq t$ and $X_1(0) \leqq x \leqq X_2(0)$ respectively. Then[1]

$$\lim_{x \to X_i(t) \pm 0} \frac{u(x,t) - u(X_i(t),t)}{x - X_i(t)} = v_i(t) \equiv \lim_{x \to X_i(t) \pm 0} \frac{\partial u}{\partial x} \qquad (i = 1,2), \tag{S2.1.3}$$

where the existence of the limit on the left is a consequence of the existence of $v_i(t)$.

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[1] The sign + is used for $i = 1$, and − for $i = 2$.

In fact, for $X_1(t) < x < X_2(t)$ and $t > 0$ the fundamental formula (1.2.3) of Part 2, Chapter I, gives

$$u(x,t) = \int_{X_1(0)}^{X_2(0)} \phi(\xi)\, E(x-\xi,t)\, d\xi$$

$$+ \sum_{i=1}^{2} (-1)^i \int_0^t \Big[v_i(\tau)\, E(x - X_i(\tau), t-\tau) \tag{S2.1.4}$$

$$- f_i(\tau) \left(\frac{\partial}{\partial \xi} - \dot{X}_i(\tau) \right) E(x - X_i(\tau), t-\tau) \Big]\, d\tau.$$

Here $E(x,t)$ is the fundamental solution of the heat equation, defined in agreement with (0.2.4), and

$$v_i(t) = \lim_{x \to X_i(t) + (-1)^{i+1} 0} \frac{\partial}{\partial x}\, u(x,t). \tag{S2.1.5}$$

Noting that

$$(-1)^i \left[\frac{\partial}{\partial \xi} - \dot{X}_i(\tau) \right] E(x - X_i(\tau), t-\tau) \tag{S2.1.6}$$

$$= \frac{1}{2} \frac{d}{d\tau} \operatorname{erfc} \frac{(-1)^i (X_i(\tau) - x)}{2\sqrt{t-\tau}},$$

we see that (S2.1.4) can be written in the form

$$u(x,t) = \int_{X_1(0)}^{X_2(0)} \phi(\xi)\, E(x-\xi,t)\, d\xi$$

$$+ \sum_{i=1}^{2} \Big[\frac{1}{2} f_i(0) \operatorname{erfc} \frac{(-1)^i (X_i(0) - x)}{2\sqrt{t}} \tag{S2.1.7}$$

$$+ \frac{1}{2} \int_0^t \dot{f}_i(\tau) \operatorname{erfc} \frac{(-1)^i (X_i(\tau) - x)}{2\sqrt{t-\tau}}\, d\tau$$

$$+ (-1)^i \int_0^t v_i(\tau)\, E(x - X_i(\tau), t-\tau)\, d\tau \Big].$$

Hence we conclude that

$$u(X_j(t), t) = \int_{X_1(0)}^{X_2(0)} \phi(\xi)\, E(X_j(t) - \xi, t)\, d\xi$$

$$+ \sum_{i=1}^{2} \Big[\frac{1}{2} f_i(0) \operatorname{erfc} \frac{(-1)^i (X_i(0) - X_j(t))}{2\sqrt{t}} + \tag{S2.1.8}$$

$$+\frac{1}{2}\int_0^t \dot{f}_i(\tau)\,\operatorname{erfc}\frac{(-1)^i(\mathrm{X}_i(\tau)-\mathrm{X}_j(t))}{2\sqrt{t-\tau}}\,d\tau$$

$$+(-1)^i\int_0^t v_i(\tau)\,E(\mathrm{X}_j(t)-\mathrm{X}_i(\tau),t-\tau)\,d\tau\Big].$$

Using the formula of finite differences, we find that

$$\frac{u(x,t)-u(\mathrm{X}_j(t),t)}{x-\mathrm{X}_j(t)}=\int_{\mathrm{X}_1(0)}^{\mathrm{X}_2(0)}\phi(\xi)\frac{\partial}{\partial x}E(z_1-\xi,t)\,d\xi$$

$$+\sum_{i=1}^{2}(-1)^i\Big[f_i(0)\,E(z_{2,i}-\mathrm{X}_i(0),t)$$

(S2.1.9)

$$+\int_0^t \dot{f}_i(\tau)\,E(z_{3,i}-\mathrm{X}_i(\tau),t-\tau)\,d\tau$$

$$+\int_0^t v_i(\tau)\frac{\partial}{\partial x}E(z_4-\mathrm{X}_i(\tau),t-\tau)\,d\tau\Big],$$

where $z_i, z_{i,j}$ $(i=1,2,3,4)$ are certain points lying between x and $\mathrm{X}_j(t)$. Every integral on the right side except the last is a heat potential of a double layer, converging uniformly for $-\infty<x<\infty$ and $t\geqq t_0$ for any value $t_0>0$. Using this and the gap theorem for the double layer heat potential, we find that the right side converges for $x\to\mathrm{X}_j(t)+(-1)^{j+1}\cdot 0$ and

$$V_j(t)=\int_{\mathrm{X}_1(0)}^{\mathrm{X}_2(0)}\phi(\xi)\frac{\partial}{\partial x}E(\mathrm{X}_j(t)-\xi,t)\,d\xi$$

$$+\sum_{i=1}^{2}(-1)^i\Big[f_i(0)\,E(\mathrm{X}_j(t)-\mathrm{X}_i(0),t)$$

(S2.1.10)

$$+\int_0^t \dot{f}_i(\tau)\,E(\mathrm{X}_j(t)-\mathrm{X}_i(\tau),t-\tau)\,d\tau$$

$$+\int_0^t v_i(\tau)\frac{\partial}{\partial x}E(\mathrm{X}_j(t)-\mathrm{X}_i(\tau),t-\tau)\,d\tau\Big]+\frac{1}{2}v_j(t).$$

Furthermore, the same result is given by differentiating (S2.1.7) with respect to x for $\mathrm{X}_1(t)<x<\mathrm{X}_2(t)$ and $t\geqq t_0$ and then taking the limit for $x\to\mathrm{X}_j(t)$. Thus the theorem is proved under the assumption of existence of the limit of the value $v_i(t)$ of the derivative $\partial u/\partial x$ at the point of the boundary $x=\mathrm{X}_i(t)$ $(i=1,2)$.

We note now that the equation $v_i(t)=V_i(t)$ $(i=1,2)$ can be con-

sidered as a system of integral equations relative to $v_i(t)$. Setting $w_i(t) = \sqrt{t}v_i(t)$ $(i = 1, 2)$, forming the integral containing ϕ, and integrating by parts, we find that $w_i(t)$ is the solution of the system of equations

$$\frac{1}{\sqrt{t}} w_j(t) = 2\sum_{i=1}^{2}(-1)^i \Big\{ [f_i(0) - \phi(X_i(0))]E(X_j(t) - X_i(0), t) + \int_0^t \dot{f}(\tau)\, E(X_j(t) - X_i(\tau), t - \tau)\, d\tau + \int_0^t v_i(\tau) \frac{\partial}{\partial x} E(X_j(t) - X_i(\tau), t - \tau)\, d\tau \Big\} + \int_{X_1(0)}^{X_2(0)} \dot{\phi}(\xi)\, E(X_j(t) - \xi, t)\, d\xi. \tag{S2.1.11}$$

The existence and uniqueness of the solution to this system is established in exactly the same way as the existence and uniqueness of the solution to the integral equation (3.3.3) of Part 2, Chapter III. Since the system (S2.1.11) is equivalent to the problem of defining u in the sense of Theorem 10 of Chapter III, Part 2, we find that the limiting values from within the domain D^t for the boundary value $\partial u/\partial x$ do in fact exist. The theorem is proved.

§2. Green's function of the double-layer boundary problem on the line and halfline.

We examine the boundary problem

$$(S2.2.1_1) \qquad a^2(x)\frac{\partial^2 u}{\partial x^2} = \left|\frac{\partial u}{\partial t}\right| \text{ for } \alpha < x < \gamma; \quad x \neq \beta; \quad t > 0;$$

$$(S2.2.1_2) \qquad u|_{x=\beta-0} = u|_{x=\beta+0}; \quad k(x)\frac{\partial u}{\partial x}\Big|_{x=\beta-0} = k(x)\frac{\partial u}{\partial x}\Big|_{x=\beta+0}; \quad t > 0;$$

$$(S2.2.1_3) \quad l_1 u = f_1(t) \text{ for } x = \alpha; \quad l_2 u = f_2(t) \text{ for } x = \beta; \quad t > 0;$$

$$(S2.2.1_4) \qquad u = \phi(x) \text{ for } t = 0; \quad \alpha < x < \gamma.$$

Here

$$a^2(x) = \begin{cases} a_1^2 = \text{const} > 0; & x < \beta \\ a_2^2 = \text{const} > 0; & x > \beta, \end{cases}$$

$$k(x) = \begin{cases} k_1 = \text{const} > 0; & x < \beta \\ k_2 = \text{const} > 0; & x > \beta, \end{cases} \tag{S2.2.1_5}$$

and l_1 and l_2 are linear differential operators of first order

$$l_i = A_i \frac{\partial}{\partial x} + B_i, \tag{S2.2.2}$$

corresponding to boundary conditions of first, second or third order, such that if $A_i = 0$ then $B_i = 0$; if $A_i = 1$, then $B_i = 0$ or $B_i = (-1)^i$.

The Green's function of the problem (S2.2.1$_i$) $(i = 1, \cdots, 5)$, as for the single-layer problem, is a solution to the equation (S2.2.1$_1$), defined for $t > \tau$, satisfying homogeneous boundary conditions (S2.2.1$_3$), initial conditions (S2.2.1$_4$) with $\phi = \delta(x - \xi)$,where $\delta(x)$ is a Dirac δ-function, and conditions conjugate to (S2.2.1$_2$). Here ξ is any point of the interval (α, γ) different from β.

It is easily seen that $g(x, t, \xi, \tau)$, considered as a function of ξ and τ for fixed x and t, satisfies the conjugate equation

$$a^2(\xi) \frac{\partial^2 g}{\partial \xi^2} + \frac{\partial g}{\partial \tau} = 0; \quad \alpha < \xi < \gamma; \quad \xi \neq \beta; \quad \tau < t, \tag{S2.2.3$_1$}$$

boundary and initial conditions

$$l_1 g|_{\xi=\alpha} = 0; \quad l_2 g|_{\xi=\gamma} = 0; \quad g|_{\tau=t} = \delta(x - \xi) \tag{S2.2.3$_2$}$$

and conditions of agreement

$$a^2 \mu g|_{\xi=\beta-0} = a^2 \mu g|_{\xi=\beta+0}; \quad a^2 \frac{\partial g}{\partial \xi}\bigg|_{\xi=\beta-0} = a^2 \frac{\partial g}{\partial \xi}\bigg|_{\xi=\beta+0}. \tag{S2.2.3$_3$}$$

Here

$$\mu(\xi) = \begin{cases} k_2; & \xi < \beta \\ k_1; & \xi > \beta. \end{cases} \tag{S2.2.3$_4$}$$

This assertion is proved in exactly the same way as the analogous assertion for the homogeneous problem (see for example [179]).

From the equations (S2.2.3$_i$) $(i = 1, 2, 3, 4)$ it follows that the solution to the problem (S2.2.1$_i$) $(i = 1, \cdots, 5)$ can be expressed in terms of Green's function and the boundary functions $f_i(t)$ and $\phi(x)$ in the usual way, so that in the integral representation of the solution the conjugate conditions do not enter explicitly.

We will call the Green's function a fundamental solution to the double-layer problem, if it is constructed for an unbounded line ($\alpha = -\infty$, $\gamma = +\infty$). In this case the boundary conditions will be replaced by the condition of boundedness of g at infinity.

We construct the fundamental solution to the double-layer problem starting with $\beta = 0$ and the Green's functions of the first and second boundary problems on the axis $x > 0$, and starting with $\beta = 1$ and some function conjugate to it in the sense of definition (0.2.11), A_5.

It is obvious that each of these functions depends only on the difference $t - \tau$. Therefore we will write $g(x, \xi, t)$ in place of $g(x, t, \xi, \tau)$, setting $\tau = 0$. The construction is carried out by means of the Laplace-Carson transform. The correspondence between the representation and the original is denoted by the sign $\doteq$.

A. **The fundamental solution of the double-layer problem** [137]. Let $G(x, \xi, p) \doteq g(x, \xi, t)$. Then

$$a^2(x)\frac{d^2G}{\partial x^2} = pG - p\delta(x-\xi); \quad -\infty < x < \infty; \quad x \neq 0;$$

$$\text{(S2.2.4)} \quad G|_{x=-0} = G|_{x=+0}; \quad k(x)\frac{dG}{\partial x}\bigg|_{x=-0} = k(x)\frac{dG}{dx}\bigg|_{x=+0};$$

$$|G| < \infty \quad \text{for} \quad x \to \infty; \quad \xi = \text{const}; \quad p = \text{const}.$$

The first of these equations is equivalent to the system

$$a^2(x)\frac{d^2G}{dx^2} = pG; \quad -\infty < x < \infty; \quad x \neq 0; \quad x \neq \xi;$$

$$\text{(S2.2.5)}$$

$$G|_{x=\xi-0} = G|_{x=\xi+0}; \quad \frac{dG}{dx}\bigg|_{x=\xi-0} - \frac{dG}{dx}\bigg|_{x=\xi+0} = \frac{p}{a^2(\xi)}.$$

On integration, we obtain

$$\text{(S2.2.6)} \quad G(x,\xi,p) = \begin{cases} \dfrac{\sqrt{p}}{2a_1}\left[\exp\left(-|x-\xi|\dfrac{\sqrt{p}}{a_1}\right) + \delta\exp\left(-|x-\xi|\dfrac{\sqrt{p}}{a_1}\right)\right. & \\ \qquad\qquad \text{for} \quad -\infty < x, \quad \xi < 0; & \\ \dfrac{\sqrt{p}}{2a_1}(1+\delta)\exp\left(-\left(\dfrac{x}{a_2} - \dfrac{\xi}{a_1}\right)\sqrt{p}\right) \quad \text{for} \quad -\infty < \xi < 0 < x; & \\ \dfrac{\sqrt{p}}{2a_2}(1-\delta)\exp\left(-\left|\dfrac{x}{a_1} - \dfrac{\xi}{a_2}\right|\sqrt{p}\right) \quad \text{for} \quad -\infty < x < 0 < \xi; & \\ \dfrac{\sqrt{p}}{2a_2}\left[\exp\left(-|x-\xi|\dfrac{\sqrt{p}}{a_2}\right) - \delta\exp\left(-|x+\xi|\dfrac{\sqrt{p}}{a_2}\right)\right] & \\ \qquad\qquad \text{for} \quad 0 < x; \quad \xi < \infty. & \end{cases}$$

Here

(S2.2.7) $$\delta = \frac{1-\lambda}{1+\lambda}; \qquad \lambda = \frac{k_2 a_1}{k_1 a_2}.$$

Using the known relation

(S2.2.8) $$\sqrt{p}\exp(-\alpha\sqrt{p}) \div (\sqrt{\pi t})^{-1}\exp\left(-\frac{\alpha^2}{4t}\right) \equiv 2E(\alpha, t),$$

we find that

(S2.2.9) $$g(x,\xi,t) = \begin{cases} E(x-\xi, a_1^2 t) + \delta E(x+\xi, a_1^2 t) & \text{for } -\infty < x, \xi < 0; \\ \dfrac{1+\delta}{a_1} E\left(\dfrac{x}{a_2} - \dfrac{\xi}{a_1}, t\right) & \text{for } -\infty < \xi < 0 < x < \infty; \\ \dfrac{1-\delta}{a_2} E\left(\dfrac{x}{a_1} - \dfrac{\xi}{a_2}, t\right) & \text{for } -\infty < x < 0 < \xi < \infty; \\ E(x-\xi, a_2^2 t) - \delta E(x+\xi, a_2^2 t) & \text{for } 0 < x, \xi < \infty. \end{cases}$$

Let g^* be the function conjugate to g, so that

(S2.2.10) $$\frac{\partial g^*}{\partial \xi} = -\frac{\partial g}{\partial x}.$$

Normalizing g^* by requiring that it vanish at infinity, we obtain

(S2.2.10*) $$g^*(x,\xi,t) = \begin{cases} E(x-\xi, a_1^2 t) - \delta E(x+\xi, a_1^2 t) & \text{for } -\infty < x, \xi < 0; \\ \dfrac{1+\delta}{a_2} E\left(\dfrac{x}{a_2} - \dfrac{\xi}{a_1}, t\right) & \text{for } -\infty < \xi < 0 < x < \infty; \\ \dfrac{1-\delta}{a_1} E\left(\dfrac{x}{a_1} - \dfrac{\xi}{a_2}, t\right) & \text{for } -\infty < x < 0 < \xi < \infty; \\ E(x-\xi, a_2^2 t) + \delta E(x+\xi, a_2^2 t) & \text{for } 0 < x, \xi < \infty. \end{cases}$$

B. **The Green's function of a double layer for the first boundary problem on the halfline.** Again let $G(x,\xi,p) \div g(x,\xi,t)$. We have

$$a^2(x)\frac{d^2G}{dx^2} = pG - p\delta(x-\xi); \quad 0 < x < \infty; \quad x \neq 1;$$

(S2.2.11) $$G|_{x=1-0} = G|_{x=1+0}; \quad k_1 \frac{dG}{dx}\bigg|_{x=1-0} = k_2 \frac{dG}{dx}\bigg|_{x=1+0};$$

$$G|_{x=0} = 0; \quad |G| < \infty \quad \text{for } x \to \infty; \quad \xi = \text{const.}$$

Integration gives

$$G(x,\xi,p) = \begin{cases} \dfrac{\sqrt{p}}{2a_1} F \cdot \left\{ \exp\left(-\dfrac{|x-\xi|}{a_1}\sqrt{p} \right) - \exp\left(-\dfrac{(x+\xi)}{a_1}\sqrt{p} \right) \right. \\ \qquad + \delta \exp\left(-\dfrac{2-x-\xi}{a_1}\sqrt{p} \right) \\ \qquad \left. - \delta \exp\left(-\dfrac{2-|x-\xi|}{a_1}\sqrt{p} \right) \right\} \\ \qquad \text{for } 0 < x, \xi < 1; \\ (1+\delta)\dfrac{\sqrt{p}}{2a_1} F \cdot \left\{ \exp\left(-\left[\dfrac{1-\xi}{a_1} + \dfrac{x-1}{a_2} \right]\sqrt{p} \right) \right. \\ \qquad \left. - \exp\left(-\left[\dfrac{x-1}{a_2} - \dfrac{1+\xi}{a_1} \right]\sqrt{p} \right) \right\} \\ \qquad \text{for } 0 < \xi < 1 < x < \infty; \\ (1-\delta)\dfrac{\sqrt{p}}{2a_2} F \cdot \left\{ \exp\left(-\left[\dfrac{1-x}{a_1} + \dfrac{\xi-1}{a_2} \right]\sqrt{p} \right) \right. \\ \qquad \left. - \exp\left(-\left[\dfrac{x+1}{a_1} + \dfrac{\xi-1}{a_2} \right]\sqrt{p} \right) \right\} \\ \qquad \text{for } 0 < x < 1 < \xi < \infty; \\ \dfrac{\sqrt{p}}{2a_2} F \cdot \left\{ \exp\left(-\dfrac{|x-\xi|}{a_2}\sqrt{p} \right) \right. \\ \qquad - \exp\left(-\left[\dfrac{x+\xi-2}{a_2} + \dfrac{2}{a_1} \right]\sqrt{p} \right) \\ \qquad + \delta \exp\left(-\left[\dfrac{2}{a_1} - \dfrac{|x-\xi|}{a_2} \right]\sqrt{p} \right) \\ \qquad \left. - \delta \exp\left(-\dfrac{x+\xi-2}{a_2}\sqrt{p} \right) \right\} \\ \qquad \text{for } 1 < x, \xi < \infty. \end{cases} \tag{S2.2.12}$$

Here

$$F = \left[1 + \delta \exp\left(-\frac{2\sqrt{p}}{a_1} \right) \right]^{-1} \tag{S2.2.13}$$

and δ is defined in agreement with (S2.2.7).

Expanding F in a series with respect to powers of $\delta \exp(-2\sqrt{p}/a_1)$

and using (S2.2.8), we find that

$$g(x,\xi,t) = \begin{cases} \displaystyle\sum_{n=0}^{\infty} (-1)^n \delta^n \{ E(|x-\xi| + 2n, a_1^2 t) - E(x+\xi+2n, a_1^2 t) \\ \qquad + \delta E(2n+2-x-\xi, a_1^2 t) \\ \qquad - \delta E(2n+2-|x-\xi|, a_1^2 t) \} \\ \qquad \text{for } 0 < x, \xi < 1; \\ \displaystyle\sum_{n=0}^{\infty} (-1)^n \delta^n (1+\delta) a_1^{-1} \cdot \left\{ E\left(\frac{x-1}{a_2} + \frac{2n+1-\xi}{a_1}, t\right) \right. \\ \qquad \left. - E\left(\frac{x-1}{a_2} + \frac{2n+1+\xi}{a_1}, t\right) \right\} \\ \qquad \text{for } 0 < \xi < 1 < x < \infty; \\ \displaystyle\sum_{n=0}^{\infty} (-1)^n \delta^n (1-\delta) a_2^{-1} \left\{ E\left(\frac{\xi-1}{a_2} + \frac{2n+1-x}{a_1}, t\right) \right. \\ \qquad \left. - E\left(\frac{\xi-1}{a_2} + \frac{2n+1+x}{a_1}, t\right) \right\} \\ \qquad \text{for } 0 < x < 1 < \xi < \infty; \\ \displaystyle\sum_{n=0}^{\infty} (-1)^n \delta^n a_2^{-1} \left\{ E\left(\frac{|x-\xi|}{a_2} + \frac{2n}{a_1}, t\right) \right. \\ \qquad - E\left(\frac{x+\xi-2}{a_2} + \frac{2n+2}{a_1}, t\right) \\ \qquad + \delta E\left(\frac{|x-\xi|}{a_2} + \frac{2n+2}{a_1}, t\right) \\ \qquad \left. - \delta E\left(\frac{x+\xi-2}{a_2} + \frac{2n}{a_1}, t\right) \right\} \\ \qquad \text{for } 1 < x, \xi < \infty. \end{cases} \tag{S3.3.14}$$

It is easily seen that the function $g^*(x,\xi,t)$, conjugate to $g(x,\xi,t)$ and normalized by the condition

$$\frac{\partial}{\partial \xi} g(x,0,t) = 0, \tag{S2.2.15}$$

has the form

$$
g^* = \begin{cases}
\displaystyle\sum_{n=0}^{\infty} (-1)^n \delta^n \{ E(|x-\xi| + 2n, a_1^2 t) + E(x+\xi+2n, a_1^2 t) \\
\qquad - \delta E(2n+2-|x-\xi|, a_1^2 t) \\
\qquad\qquad - \delta E(2n+2-x-\xi, a_1^2 t) \} \\
\qquad\qquad \text{for } 0 < x, \xi < 1; \\
\displaystyle\sum_{n=0}^{\infty} (-1)^n \delta^n (1+\delta) a_2^{-1} \left\{ E\left(\frac{x-1}{a_2} + \frac{2n+1-\xi}{a_1}, t\right) \right. \\
\qquad\qquad \left. + E\left(\frac{x-1}{a_2} + \frac{2n+1+\xi}{a_1}, t\right) \right\} \\
\qquad\qquad \text{for } 0 < \xi < 1 < x < \infty; \\
\displaystyle\sum_{n=0}^{\infty} (-1)^n \delta^n (1-\delta) a_1^{-1} \left\{ E\left(\frac{\xi-1}{a_2} + \frac{2n+1-x}{a_1}, t\right) \right. \\
\qquad\qquad \left. + E\left(\frac{\xi-1}{a_2} + \frac{2n+1+x}{a_1}, t\right) \right\} \\
\qquad\qquad \text{for } 0 < x < 1 < \xi < \infty; \\
\displaystyle\sum_{n=0}^{\infty} (-1)^n \delta^n a_2^{-1} \left\{ E\left(\frac{|x-\xi|}{a_1} + \frac{2n}{a_1}, t\right) \right. \\
\qquad + E\left(\frac{x+\xi-2}{a_2} + \frac{2n+2}{a_1}, t\right) \\
\qquad + \delta E\left(\frac{|x-\xi|}{a_2} + \frac{2n+2}{a_1}, t\right) \\
\qquad\qquad \left. + \delta E\left(\frac{x+\xi-2}{a_2} + \frac{2n}{a_1}, t\right) \right\} \text{ for } 1 < x, \xi < \infty,
\end{cases}
\tag{S2.2.16}
$$

C. **The Green's function of a double layer for the second boundary problem on the halfline.** Omitting the calculations, which are completely analogous to those of the previous paragraphs, we give the final result:

$$
\begin{aligned}
g(x,\xi,t) &= E(x-\xi, a_1^2 t) + E(x+\xi, a_1^2 t) \\
&\quad + \sum_{n=1}^{\infty} \delta^n [E(x-\xi-2n, a_1^2 t) \\
&\qquad + E(x-\xi+2n, a_1^2 t) + E(x+\xi-2n, a_1^2 t) \\
&\qquad + E(x+\xi+2n, a_1^2 t)] \text{ for } 0 < x, \xi < 1;
\end{aligned}
$$

$$g(x,\xi,t) = \begin{cases} (1+\delta)a_1^{-1}\sum_{n=0}^{\infty}\delta^n\left[E\left(\frac{x-1}{a_2}+\frac{1-\xi+2n}{a_1},t\right)\right. \\ \qquad\qquad \left. + E\left(\frac{x-1}{a_2}+\frac{1+\xi+2n}{a_1},t\right)\right] \\ \qquad\qquad \text{for } 0<\xi<1<x<\infty; \\ (1-\delta)a_1^{-1}\sum_{n=0}^{\infty}\delta^n\left[E\left(\frac{\xi-1}{a_2}+\frac{1-x+2n}{a_1},t\right)\right. \\ \qquad\qquad \left. + E\left(\frac{\xi-1}{a_2}+\frac{1+x+2n}{a_1},t\right)\right] \\ \qquad\qquad \text{for } 0<x<1<\xi<\infty; \\ E(x-\xi,a_2^2t)-\delta E(x+\xi-2,a_2^2t) \\ \quad +(1-\delta)a_2^{-1}E\left(\frac{x+\xi-2}{a_2}+\frac{2}{a_1},t\right) \\ \quad +a_2^{-1}(1-\delta)\sum_{n=1}^{\infty}\delta^n\left[E\left(\frac{x+\xi-2}{a_2}+\frac{2n}{a_1},t\right)\right. \\ \qquad\qquad \left. + E\left(\frac{x+\xi-2}{a_2}+\frac{2n+2}{a_1},t\right)\right] \\ \qquad\qquad \text{for } 1<x,\ \xi<\infty. \end{cases} \tag{S2.2.17}$$

Finally, the function $g^*(x,\xi,t)$, conjugate to $g(x,\xi,t)$ and normalized by the condition

$$g(x,0,t)=0, \tag{S2.2.18}$$

has the form

$$\begin{aligned} g^*(x,\xi,t) = {} & E(x-\xi,a_1^2t)-E(x+\xi,a_1^2t) \\ & +\sum_{n=1}^{\infty}\delta^n[E(x-\xi-2n,a_1^2t)-E(x+\xi-2n,a_1^2t) \\ & \qquad +E(x-\xi+2n,a_1^2t)-E(x+\xi+2n,a_1^2t)] \\ & \qquad\qquad \text{for } 0<x,\ \xi<1; \end{aligned}$$

$$
g^*(x,\xi,t) = \begin{cases}
a_2^{-1}(1+\delta)\sum_{n=0}^{\infty}\delta^n\left[E\left(\frac{x-1}{a_2}+\frac{1-\xi+2n}{a_1},t\right)\right. \\
\qquad\qquad \left. - E\left(\frac{x-1}{a_2}+\frac{1+\xi+2n}{a_1},t\right)\right] \\
\qquad\qquad \text{for } 0<\xi<1<x<\infty; \\
a_1^{-1}(1-\delta)\sum_{n=0}^{\infty}\delta^n\left[E\left(\frac{\xi-1}{a_2}+\frac{1-x+2n}{a_1},t\right)\right. \\
\qquad\qquad \left. - E\left(\frac{\xi-1}{a_2}+\frac{1+x+2n}{a_1},t\right)\right] \\
\qquad\qquad \text{for } 0<x<1<\xi<\infty; \\
E(x-\xi,a_2^2t)+\delta E(x+\xi-2,a_2^2t) \\
\quad -(1-\delta)a_2^{-1}E\left(\frac{x+\xi-2}{a_2}+\frac{2}{a_1},t\right) \\
\quad -a_2^{-1}(1-\delta)\sum_{n=1}^{\infty}\delta^n\left[E\left(\frac{x+\xi-2}{a_2}+\frac{2n}{a_1},t\right)\right. \\
\qquad\qquad \left. + E\left(\frac{x+\xi-2}{a_2}+\frac{2n+2}{a_1},t\right)\right] \\
\qquad\qquad \text{for } 1<x,\xi<\infty.
\end{cases}
$$

(S2.2.19)

SUPPLEMENT III

NUMERICAL ILLUSTRATIONS

§1. Example of the numerical solution of the integral equations of the Stefan problem[1)]

We will now illustrate the solution of the single-phase Stefan problem by the method of integral equations, using the following problem as an example.

We consider a problem of heat transfer in a medium which is initially in a two-phase state, and occupies the layer $0 < x < 1$; we assume that the phase 2 material lies in the region $y_1(t) < x < y_2(t)$ and the phase 1 material lies outside of it. Suppose that $y_1(0) = 0.25$ and $y_2(0) = 0.75$. At the initial moment the temperature of the phase 2 material is equal to the critical temperature of the phase change, which we take to be zero, and the temperature of the phase 1 material is linear in each of the two regions in which it lies. At the boundaries $x = 0$ and $x = 1$ the temperature is maintained at the constant value -1.

It is obvious that the temperature of phase 2 will remain equal to zero at all times and the temperature of phase 1 will be symmetric with respect to the point $x = 0.5$. Therefore the problem reduces to one of finding $u(x,t)$ and $y(t)$ satisfying the conditions

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}; \quad 0 < x < y(t); \quad t > 0;$$

$$u|_{x=0} = -1; \quad u|_{t=0} = 4(x - 0.25); \quad u|_{x=y(t)} = 0; \tag{S3.1.1}$$

$$\dot{y}(t) = v(t); \quad y(0) = 0.25; \quad v \equiv \frac{\partial}{\partial x} u(y(t),t).$$

The calculation of the temperature u after determining v and y is of no interest. We limit ourselves to the calculation of v and y on the interval $(0, T)$, where T is the time interval in which phase 2 exists, defined by the equation $y(T) = .5$.

[1)] See Part 2, Chapter VIII, §3. In the following we describe methods which are oriented towards numerical hand computation. In using the ÈVSM computer, the methods of Part 2, Chapter VIII, §3 are employed.

In the case under examination the integral equations $(2.2.3_i)$ $(i = 1, 2, 3)$ assume the following form:

$$v(t) = 4\left[\operatorname{erf}\frac{y(t)+0.25}{2\sqrt{t}} - \operatorname{erf}\frac{y(t)-0.25}{2\sqrt{t}}\right]$$

$$-\frac{2}{\sqrt{\pi}}\int_0^t W(t,\tau\,|\,v(\tau),y(t),y(\tau))\frac{d\tau}{\sqrt{t-\tau}}; \tag{S3.1.2}$$

$$y(t) = 0.25 + \int_0^t v(\tau)\,d\tau,$$

where

$$W = v(\tau)\left\{\frac{y(t)-y(\tau)}{4(t-\tau)}\exp\left[-\frac{(y(t)-y(\tau))^2}{4(t-\tau)}\right]\right. \tag{S3.1.3}$$

$$\left. -\frac{y(t)+y(\tau)}{4(t-\tau)}\exp\left[-\frac{(y(t)+y(\tau))^2}{4(t-\tau)}\right]\right\}.$$

We begin the calculations with a preliminary estimate of the length T of the process, which can be done by the use of Lemma 1 of Part 2, Chapter II.

We set

$$u_1 = -\left(1 - \frac{\operatorname{erf}(x/2\sqrt{t})}{\operatorname{erf}\gamma}\right); \quad z_1 = 2\gamma\sqrt{t} \tag{S3.1.4}$$

and choose γ such that

$$\dot{z}_1 = \frac{\gamma}{\sqrt{t}} < \frac{\partial u_1}{\partial x}\bigg|_{x=z_1} = \frac{\exp(-\gamma^2)}{\sqrt{\pi t}\operatorname{erf}\gamma}. \tag{S3.1.5}$$

It is obvious that, as they are defined, u_1 and z_1 satisfy every condition of the lemma. Thus $z_1(t) \leqq y(t)$ for every $t \in (0, T)$. Calculation shows that one can take $\gamma = 0.6$. This gives $z_1(t) = 0.5$ for $t \cong 0.174$. Thus $T < 0.174$.

On the other hand, in order to determine a majorant of y it suffices for us to take

$$u_2 = 4(x - z_2(t)); \quad z_2(t) = \sqrt{0.0625 + 2t}. \tag{S3.1.6}$$

This gives $T > 0.093$. Thus

$$0.093 < T < 0.174. \tag{S3.1.7}$$

We pose the problem of determining $y(t)$ with precision 0.0025,

which corresponds to the maximum relative error of 1%. From (S3.1.7) and (S3.1.1) it follows that for this it automatically suffices to evaluate $v(t)$ with an accuracy of 0.014. This requirement can be met by choosing the mesh length appropriately.

TABLE 1.

Supporting value t	Mesh length	Method of Interpolation	Remarks
0.00001	0.00001	Linear	
0.00002			
0.00004			
0.00007			
0.00010			
0.00020	0.00005		
0.00040	0.0001		
0.0010	0.0002		
0.0025	0.0005		
0.0050			
0.010	0.0010		
			Interval 0.018—0.020 with mesh 0.001
0.020	0.0020		0.028—0.030 with mesh 0.001
0.030			0.038—0.040 with mesh 0.001
0.040			0.055—0.060 with mesh 0.0025
		Parabolic	
0.060	0.0050	3 Points	
0.080	0.01		0.070—0.080 with mesh 0.0025
0.100			0.090—0.100 with mesh 0.0025
0.125			0.120—0.125 with mesh 0.0025

We shall call the value t for which $v(t)$ is evaluated by means of an iteration process, a supporting value, and the values for which $v(t)$ is defined by means of interpolation, intermediate. In Table 1 we indicate the supporting values t, the length of the mesh for calculating $v(t)$ for supporting value t, and the method of interpolation used for evaluating $v(t)$ at intermediate values t.

The choice of the mesh length and the method of interpolation ensures the attainment of the required accuracy in every case. Here we do not examine the problem of finding the greatest permissible mesh length

or an optimal method of numerical integration. The essential requirement which we make is that of simplicity of computation on a calculator.

The integral on the right side of (S3.1.2) is calculated in the following way. The interval $(0,t)$ is divided by a point $t^* < t$ into two parts. Along the interval $0 \leqq \tau \leqq t^*$ the integral is evaluated by Simpson's rule with the mesh lengths indicated in Table 1. Along the interval (t^*,t), where Simpson's rule is inapplicable, the integral is evaluated by the formula (8.3.19). In the last column of Table 1 we indicate an additional subdivision of the integration interval which is chosen so finely that the magnitude of the oscillation of $W(t,\tau)$ on each of the intervals obtained by the subdivision is small enough to ensure that the error of computation by formula (8.3.19) does not exceed a given degree of tolerance.[2)]

The choice of the "zeroth" approximation $v_0(t_i)$ is carried out by means of graphical extrapolation for $t > 0.01$. For $0.00002 \leqq t \leqq 0.01$ we use the formula

$$v_0(t_{i+1}) = v(t_i) - \alpha \frac{v(t_{i-1}) - v(t_i)}{t_i - t_{i-1}} (t_{i+1} - t_i), \tag{S3.1.8}$$

where the correction coefficient α is introduced with the aim of calculating the decrease $|\dot{v}(t)|$. We take $0.5 \leqq \alpha \leqq 0.75$.[3)] Finally, $v_0(0.00001)$ is taken equal to $v(0) = 4$. In Table 2 and Figure 7 the results of calculating $v_n(t)$ and $y_n(t)$ for supporting value t are given. Here n is the number of iterations.

As is seen from the table, the error of iteration for evaluating $v(t)$ is not greater than 0.001 for $n \leqq 3$ for every value of t except $t = 0$, $t = 0.02$ and $t = 0.06$. For $t = 0.02$ this error is not greater than 0.001 for $n \geqq 4$, and for $t = 0.06$ it is less than 0.002 for $n \geqq 5$. Obviously for these points the graphical extrapolation for chosen t_k is less satisfactory. We note that $v(0.006) = 1.944$. Thus the absolute error equals 0.002 while the corresponding relative error equals 0.1%.

[2)] If the number of subintervals in $(0,t^*)$ is odd, then we apply Simpson's rule to the interval (t_1,t^*), where t_1 is the nearest subdivision point to $t = 0$. Within the interval $(0,t_1)$ the integral is evaluated by the trapezoidal rule.

[3)] In practice the introduction of the correction coefficient α is convenient, although the numerical value of α is generally chosen without theoretical justification. Making use of the Taylor formula for small t would be highly inconvenient because of the rapid variation of $v(t)$. In order to guarantee sufficient closeness to the actual value $v_0(t_i)$ with the aid of Taylor's formula would require differences of order higher than that which is possible with the number of points and amount of computation feasible.

TABLE 2

t	$n=0$		$n=1$		$n=2$		$n=3$		$n=4$		$n=5$		$n=6$	
	v	y	v	y	v	y	v	y	v	y	v	y	v	y
0	4	0.25		0.250000										
0.00001	4	0.250040	3.942_9	0.250040	3.943_5	0.250040	3.943_3	0.250040						
0.00002	3.915_1	0.250079	3.921_0	0.250079	3.920_9	0.250079	3.920_9	0.250079						
0.00004	3.904_2	0.250157	3.889_3	0.250157	3.889_5	0.250157	3.889_5	0.250157						
0.00007	3.874_8	0.250274	3.854_2	0.250273	3.854_4	0.250273	3.854_4	0.250273						
0.0001	3.846_1	0.250389	3.827_2	0.250389	3.828_5	0.250389	3.828_4	0.250389						
0.0002	3.793_0	0.250770	3.760_6	0.250767	3.762_6	0.250767	3.762_6	0.250767						
0.0004	3.713_3	0.251514	3.677_3	0.251509	3.673_6	0.251508	3.674_0	0.251508						
0.0010	3.540	0.253683	3.504_0	0.253671	3.509_4	0.253673	3.508_8	0.253673						
0.0025	3.304	0.258776	3.278_8	0.258759	3.281_6	0.258760	3.281_9	0.258760						
0.0050	3.092	0.266730	3.047_0	0.266675	3.061_8	0.266693	3.058_7	0.266690	3.059_0	0.266690				
0.01	2.906	0.281480	2.787_6	0.281302	2.811_2	0.281359	2.805_5	0.281347	2.806_8	0.281347				
0.02	2.615	0.308268	2.487_3	0.307806	2.517_8	0.307958	2.508_2	0.307906	2.511_1	0.307925	2.510_7	0.307925		
0.03	2.311	0.332060	2.319_2	0.332077	2.318_4	0.332077								
0.04	2.188	0.354607	2.165_8	0.354495	2.172_8	0.354533	2.169_9	0.354519	2.170_7					
0.06	2.018	0.396079	1.858_7	0.394757	1.970_5	0.395685	1.936_9	0.395394	1.948_7	0.395495	1.943_6	0.395471	1.945_0	
0.08	1.774	0.432557	1.778_6	0.423601	1.777	0.432581	1.777_4	0.432581						
0.10	1.670	0.466951	1.641_8	0.466724	1.650_5	0.466777	1.647_0	0.466754	1.647_8					
0.125	1.515	0.506209	1.519_5	0.506248	1.517_9	0.506240	1.518_7							

TABLE 3

$I_{\alpha\beta}$; $\alpha = 0.00001$, $\beta = 0.00009$			$I_{\alpha\beta}$; $\alpha = 0$, $\beta = 0.12$		
Trapezoidal formula	Simpson's formula	Formula (8.3.19)	Trapezoidal formula	Simpson's formula	Formula (8.3.19)
0.05418	0.05367	0.05367	0.30539	0.30070	0.30100

As an illustration, in Table 3 we give the error of the approximation of the value

(S3.1.9) $$I_{\alpha\beta} = \frac{2}{\sqrt{\pi}} \int_{\alpha}^{\beta} W(t, \tau \mid y_n, v_n) \frac{d\tau}{\sqrt{t-\tau}}$$

for $t = 0.0001$ for $n = 0$ and $t = 0.125$ for $n = 2$. $I_{\alpha\beta}$ is evaluated by the trapezoidal rule, by Simpson's formula and by formula (8.3.19). Here (α, β) is the largest interval along which the integration can be carried out by Simpson's rule if we use the network shown in Table 1. We thus see that the error of the approximation in calculating $v(0.0001)$ and $v(0.125)$ is not greater than 0.005. In addition the estimated approxi-

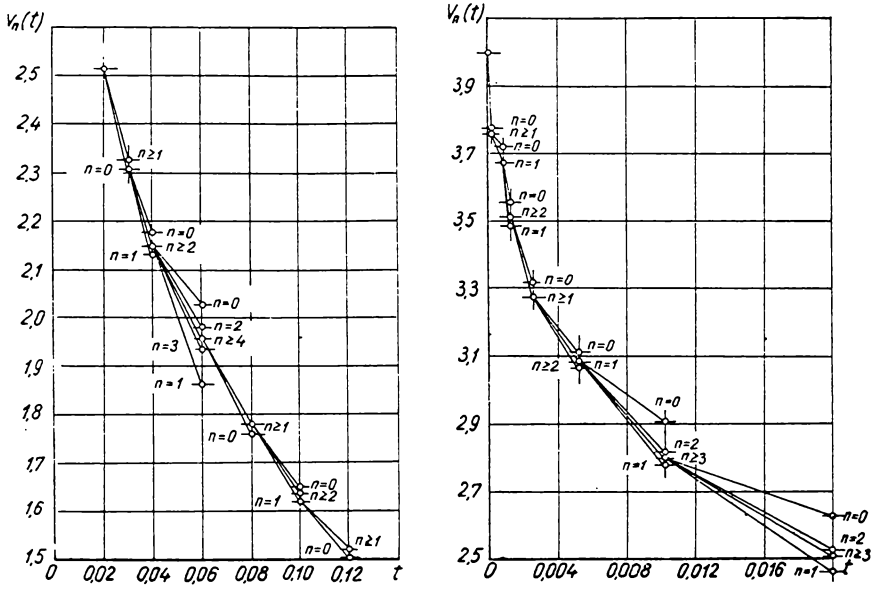

FIGURE 7.

mation error on every interval examined varies with time. But together with the estimates of the iteration error given below, this proves that the error of determination does not in fact exceed the tolerable value 0.014 on every interval $0 < t < 0.125$.

TABLE 4

A — "Exact" solution (method of integral equations)
B — Quasi-stationary approximation of Leĭbenzon
C — Modified method of Leĭbenzon

t	y_A	y_B	y_C	Remarks
0.00001	0.250040	0.250040	0.250040	$y(T) = 0.5$
0.00010	0.250389	0.250400	0.250394	
0.00100	0.253673	0.253968	0.253820	. . . . .
0.00250	0.258760	0.259808	0.259293	
0.00500	0.266690	0.269258	0.268041	$\delta T_C = \left(1 - \frac{T_B}{T_A}\right) 100$
0.01000	0.281347	0.287228	0.284637	
0.02000	0.307925	0.320156	0.315603	
0.03000	0.332077	0.350000	0.344085	
0.04000	0.354519	0.377492	0.370676	
0.06000	0.395471	0.427200	0.419298	
0.08000	0.432581	0.476990	0.441749	
0.10000	0.466754	0.512348	0.503410	
0.12500	0.506240	0.559017	0.551453	
0.12092*	0.5			
0.09375*		0.5		* Obtained by
0.10726*			0.5	linear interpolation

$\delta T_B = 22.5\%$; $\delta T_C = 10.8\%$

In Table 4 we give the "exact" solution values corresponding to the quasi-stationary approximation of Leĭbenzon and with modifications used by us in §§1 and 2 of Chapters II and III, Part 1. Namely, Leĭbenzon's quasi-stationary approximation is determined as the solution of the problem

$$d^2u/dx^2 = 0; \quad 0 < x < y(t); \quad u|_{x=0} = -1; \quad u|_{x=y(t)} = 0;$$
$$\dot{y}(t) = v(t) \equiv du/dx; \quad y(0) = 0.25, \tag{S3.1.10}$$

i.e. coinciding with the functions u_2, z_2 defined in (S3.1.6). A modification of the same method reduces to the solution of the problem

$$\partial^2 u/\partial x^2 = \partial u/\partial t; \quad 0 < x < y(\lambda); \quad u|_{x=0} = -1; \quad u|_{x=y(\lambda)} = 0;$$

(S3.1.11) $$u|_{t=0} = 4(x - 0.25)\eta(y(\lambda) - x)\eta(0.25 - x);$$

$$\dot{y}(t) = \partial u(y(\lambda), t)/\partial x|_{\lambda=t} \equiv v(t); \quad y(0) = l = 0.25.$$

Using the expression

(S3.1.12) $$g_i(x, \xi, t) = \sum_{n=-\infty}^{\infty} [E(x - \xi + 2ny(\lambda), t) + (-1)^i E(x + \xi - 2ny(\lambda), t)]$$

for the Green's functions of the first and second boundary problems on the interval $(0, y(\lambda))$, we find without difficulty that

(S3.1.13) $$\dot{y}(t) = 4\sum_{n=0}^{\infty}\left[\operatorname{erf}\frac{y(t)(2n+1) + l}{2\sqrt{t}} - \operatorname{erf}\frac{y(t)(2n+1) - l}{2\sqrt{t}}\right].$$

Equation (S3.1.13) was integrated on the BÈSM-2M by the method of Runge-Kutta using a standard program, ensuring a result with the accuracy of six figures.

In the last line of the table we find the size of the relative error of determining the final moment of the process (i.e. the moment when the boundary reaches $y = 0.5$). Assume that this moment is determined "exactly" by the method of integral equations. As we see, on changing from the quasi-stationary method to the modified quasi-stationary method we decrease the error of this determination by better than a factor of two, and hence we can use this method for empirical justification of the modified method of Leĭbenzon.

§2. Crystallization of a melt under insertion of a plate or a sphere[4)]

We examine the process of crystallization of a melt upon insertion of a plate or a sphere of more refractory material, described in dimensionless variables by the equations (5.1.1)—(5.1.7) and (6.1.7), $(6.1.8_1)$—$(6.1.8_5)$ of the second part, respectively. We pass from dimensionless to dimensioned variables by formulas (5.5.1) of Part 2, with l replaced by R_0 in the spherical case, where R_0 is the radius of the immersed sphere, and $u = (T - T_k)/(T_0 - T_k)$ by $u = x(T - T_k)/(T_0 - T_k)$.

The values of $w(t)$, $f(t)$ and $y(t)$ are determined, in the case of an

[4)] See Part 2, Chapter V, §§1 and 2; Part 2, Chapter VI, §1 and Part 1, Chapter II, §3.

immersed plate, as the solution of the system of integral equations (5.2.6), (5.2.7) of Part 2, and in the case of an immersed sphere, as the solution of the system (6.1.17), (6.1.19) and (6.1.20) of Part 2.

The numerical algorithm which we use is identical to that described earlier (see Part 2, Chapter I, §6A), except that the values w, f and y on the interval $0 \leqq t \leqq t_1$ obey the identities

$$\text{(S3.2.1)} \qquad w(t) \equiv w(0); \quad f(t) \equiv f(0); \quad y(t) \equiv 1 - \alpha t + 2\beta w(0)\sqrt{t}.$$

Here $w(0)$ is the negative root of equation (6.1.27), and $f(0)$ is the root of equation (6.1.25). This corresponds to the case of an immersed sphere. For an immersed plate the computation of $f(t)$ is unnecessary.

The integrals encountered in the evaluation are of the form

$$\text{(S3.2.2)} \qquad J_1 = \int_\alpha^\gamma \phi(t,\tau)\frac{d\tau}{\sqrt{\tau}}; \quad J_2 = \int_\alpha^\gamma \phi(t,\tau)\frac{d\tau}{\sqrt{\tau}\sqrt{t-\tau}},$$

where $\phi(t,\tau)$ is continuous on the interval (α,γ), and are evaluated by the general formula of Simpson, obtained with the aid of a quadratic interpolation of $q(t,\tau)$. This gives

$$J_1 = 2\sqrt{\gamma}\left(a + \frac{1}{3}b\gamma + \frac{1}{5}c\gamma^2\right) - 2\sqrt{\alpha}\left(a + \frac{1}{3}b\alpha + \frac{1}{5}c\alpha^2\right);$$

$$\text{(S3.2.3}_1\text{)} \qquad \begin{aligned} J_2 = {} & \left(2a + bt + \frac{3}{4}ct^2\right)\cdot\left(\arcsin\sqrt{\frac{\gamma}{t}} - \arcsin\sqrt{\frac{\alpha}{t}}\right) \\ & + (b + ct)\left(\sqrt{\alpha}\sqrt{t-\alpha} - \sqrt{\gamma}\sqrt{t-\gamma}\right) \\ & + \frac{c}{4}\left(\sqrt{\gamma}\sqrt{t-\gamma}(t-2\gamma) - \sqrt{\alpha}\sqrt{t-\alpha}(t-2\alpha)\right), \end{aligned}$$

where

$$\text{(S3.2.3}_2\text{)} \qquad \begin{aligned} a &= \frac{1}{4h^2}\left[\gamma(\alpha+\gamma)\phi(\alpha) - 4\alpha\gamma\phi(\beta) + \alpha(\alpha+\gamma)\phi(\gamma)\right]; \\ b &= -\frac{1}{4h^2}\left[(3\gamma+\alpha)\phi(\alpha) - 4(\alpha+\gamma)\phi(\beta) + (3\alpha+\gamma)\phi(\gamma)\right]; \\ c &= \frac{1}{4h^2}\left[2\phi(\alpha) - 4\phi(\beta) + 2\phi(\gamma)\right]; \\ h &= \frac{\gamma-\alpha}{2}; \quad \beta = \frac{\alpha+\gamma}{2}; \quad \phi(\alpha) \equiv \phi(t,\alpha), \cdots, \phi(\gamma) = \phi(t,\gamma). \end{aligned}$$

For the numerical solution of the problem on the machine BÈSM-2M a program was composed, permitting the calculation without passage to the external memory. In this connection we note the following. As is seen from the description of the algorithm, the size of the calculation increases strongly with the increase of time t, since in passing from t_k to t_{k+1} an integral of the form

$$\int_0^{t_k} F(t_k, \tau, w(\tau), y(t_k), y(\tau))\, d\tau$$

must again be computed. During the calculation the memory of the machine must store the values of w, f and y for every value t_k; this imposes a limitation on the length of the time interval on which the calculation can be conducted without passage to external memory, for a required degree of accuracy. The composition of the program permits a calculation for 545 and 344 values t_k on the line and respectively for the sphere. In the printout we are given the dimensioned time τ and, corresponding to it, the dimensionless y, w, and, in the sphere, f.

The moment at which the iteration cycle is ended is fixed with the aid of the inequalities $|y_{n+1}(t_k) - y_n(t_k)| < \epsilon_y$, $|w_{m+1}(t_k) - w_m(t_k)| < \epsilon_w$ and $|f_{p+1}(t_k) - f_p(t_k)| < \epsilon_f$. Iterations for given k are continued until all of these inequalities are fulfilled simultaneously. The values ϵ_y, ϵ_w and ϵ_f are estimated with the help of numerical tests. Thus to evaluate y with an accuracy of .001 we set ϵ_y, ϵ_w and ϵ_f roughly equal to 10^{-5}. Too small a choice of these numbers ϵ is not practical.

The accuracy of the result is controlled by repeated execution of the calculation with chosen mesh points t_k. The process is one not involving successive iterations (i.e. calculations for which it is assumed that $y \equiv y_0$, $w \equiv w_0$, $f \equiv f_0$), and in further calculations we use these iterates (i.e. with the values arising from the algorithm described in Part 2, Chapter I, §6). Noniterative calculations are conducted with constant mesh, and iterative calculations with variable mesh. As an illustration, in Table 5 we give $y_{\max}$ (= the maximum value of y), the moment $\tau_{\max}$ of its attainment, and the moment τ_k of termination of the process; these are found in the linear case for $q = 10^6 \mathrm{J/m^2 sec}$ and $l = 0.02\mathrm{m}$. (The remaining characteristics are in Table 7, in the column for iron.) Variants A and B differ in the choice of the sequence of values t_k. In Table 6 we show the sequences of mesh lengths taken in these cases.[5)]

[5)] The choice of the mesh is made by the machine without changing the basic program determined by the chosen system. Essentially our aim is the step-by-step enlargement

TABLE 5

Characteristic form of the calculation	τ_{max}, sec.	y_{max}	τ_k, sec.	Machine time spent on calculation
Without iteration $\Delta t = 2^{-3}$	38.98	1.873	129.9	3′30″
Without iteration $\Delta t = 2^{-4}$	40.60	1.836	131.5	9′45″
Without iteration $\Delta t = 2^{-5}$	41.41	1.814	132.4	36′30″
Without iteration $\Delta t = 2^{-6}$	42.22	1.800	130.7	1^{h}24′10″
With iteration Variant A	41.93	1.795	130.8	
With iteration Variant B	42.82	1.798	131.3	

As we see from Table 5, the magnitude y_{max} is determined with an accuracy of 0.01. The same accuracy is obtained for y at all times. The moment of attainment of the maximum of y is fixed with equivalent accuracy, which is roughly 1% of τ_{max}. The moment of completion of the process is fixed with great relative accuracy.[6)]

Below we give the results of calculation for the dynamic crystallization of a melt of carbon under insertion in it of a plate and sphere of iron and molybdenum. The thermal characteristics of the resulting carbon, of the iron and of the molybdenum, are given in Table 7. The calculation is carried out for $T_0 = 293°$ K for $l = 0.02$m in the linear case and $R_0 = 0.005$, 0.01 and 0.05m in the spherical case. (In the linear case the calculation is carried out only for iron; in the spherical case, for iron and molybdenum.) The size of $10^{-6} \cdot q$ is taken as $= 0.06$, 0.12, 0.36, 0.60 and 1.00J/m^{3}sec.

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of the step size in the initial calculation, and the successive reduction of the distance from the boundary to the point $y = 1$, an indication of the termination of the melting process of the crust which is crystallized from the melt.

Tables 5 and 6 refer to the case of immersion of a plate.

[6)] An analogous result can be obtained for the spherical problem.

TABLE 6

Variant A			Variant B		
Mesh Range	Number of meshes	Δt	Mesh Range	Number of meshes	Δt
1—22	22	2^{-12}	1—8	8	2^{-13}
23—51	28	2^{-11}	9—20	12	2^{-12}
52—65	15	2^{-10}	21—43	23	2^{-11}
66—124	59	2^{-9}	44—61	18	2^{-10}
125—149	25	2^{-8}	62—80	19	2^{-9}
150—186	37	2^{-7}	81—102	22	2^{-8}
187—218	32	2^{-6}	103—143	41	2^{-7}
219—344	126	2^{-5}	144—170	27	2^{-6}
345	1	2^{-6}	171—283	113	2^{-5}
346	1	2^{-7}	284—309	26	2^{-6}
347	1	2^{-8}	310—332	23	2^{-7}
			333—339	7	2^{-8}
			340—344	5	2^{-9}
			345—348	4	2^{-10}

TABLE 7

Parameter	Carbon phase $i = 2$	Iron $i = 1$	Molybdenum $i = 1$
T_k-°K	1803	1803	2893
k-J/m · sec · grad	33	70	109
ρ-kg/m^3	7200	7050	10100
C-J/kg · grad	$0.69 \cdot 10^3$	$0.645 \cdot 10^3$	$0.31 \cdot 10^3$
γ-J/kg.	$275 \cdot 10^3$	$275 \cdot 10^3$	$209 \cdot 10^3$

The results of determining y_{max}, τ_{max} and τ_k are given in Tables 8 and 9.

Parallel to the solution of the problem by this precise treatment, one may use an approximate treatment proposed by M. Ja. Antimirov (see Part 1, Chapter I, §3, and [3]). In the case of a melt the solution is determined by the equations (1.3.14), (1.3.10) and (1.3.11)—(1.3.13) of Part 1. In the case of a sphere the solution to the problem was carried

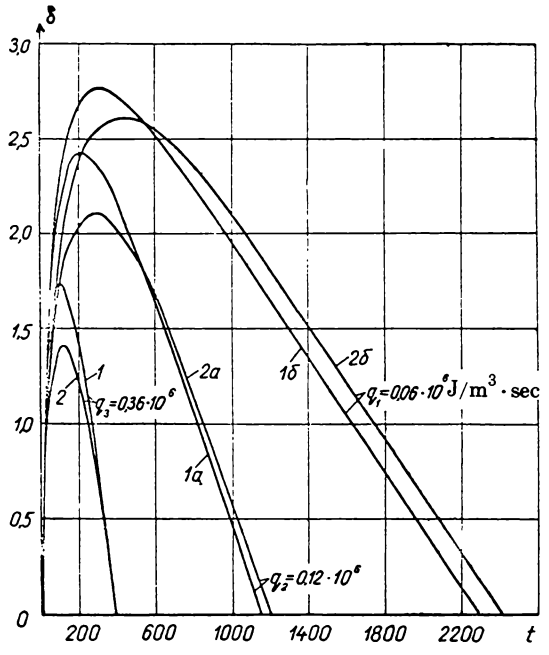

FIGURE 8. Case of an Iron Plate.
1—"Exact" solution; 2—quasi-stationary approximation.

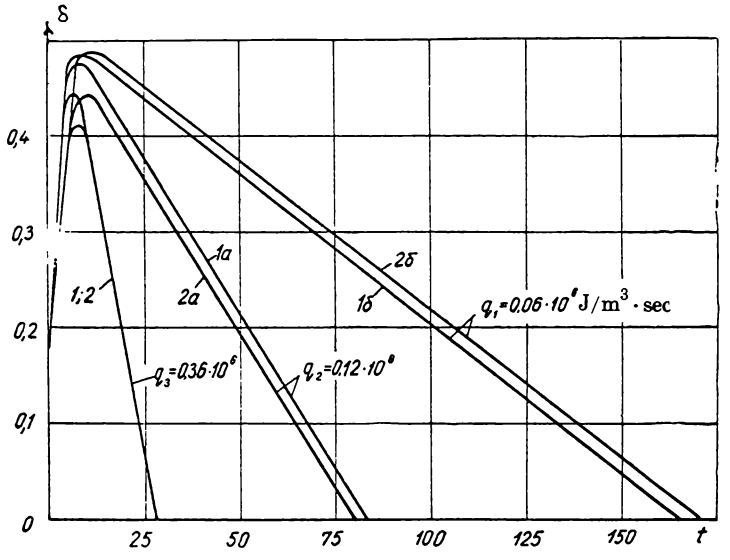

FIGURE 9. Molybdenum Sphere, $R = 0.01$ mm.
1—"Exact" solution; 2—quasi-stationary approximation.

TABLE 8

Iron plate; $l = 0.02$m

$10^{-6} \cdot q$ J/m^2 · sec	y_{max}	τ_{max} sec	τ_k sec	$10^{-6} \cdot q$ J/m^2 · sec	y_{max}	τ_{max} sec	τ_k sec
0.06	3.61	440	2420	0.60	2.10	70	218
0.12	3.12	275	1187	1.00	1.80	42	111
0.36	2.42	110	374				

TABLE 9

Sphere of molybdenum and iron

R_0, m	$q \cdot 10^{-6}$ J/m^2 · sec	Molybdenum			Iron		
		y_{max}	τ_{max} sec	τ_k sec	y_{max}	τ_{max} sec	τ_k sec
0.005	1.00	1.387	0.8	5.0	1.437	1.3	6.5
	0.60	1.417	1.1	8.4	1.517	1.6	10.8
	0.36	1.440	1.4	14.0	1.552	2.1	18.0
	0.12	1.469	1.8	42.0	1.600	2.9	54.0
	0.06	1.493	2.2	84.0	1.631	3.6	108
0.01	1.00	1.338	2.3	10.1	1.413	3.5	13.0
	0.60	1.374	3.1	16.8	1.455	4.6	21.6
	0.36	1.405	4.0	28.0	1.504	6.2	36.0
	0.12	1.440	6.0	84.0	1.574	9.0	108
	0.06	1.488	7.5	168	1.601	12.0	216
0.05	1.00	1.191	19.0	58.0	1.211	25.0	76.0
	0.60	1.243	28.0	95.0	1.275	40.0	127
	0.36	1.290	42.0	153	1.343	60.0	208
	0.12	1.380	76.0	427	1.465	115	568
	0.06	1.421	108	845	1.528	168	1105

out by Antimirov by integration of a system of ordinary differential equations.

The dependence of the thickness of the crust on the time is found for the "exact" solution of the problem in the quasi-stationary treatment represented in Figures 8 and 9. We see that after the maximum thickness is attained, the crust curve $x = y(t)$ turns out to be essentially linear, and the tangent of its angle of inclination coincides with the

corresponding value α, defined by (5.5.1) of Part 2. Thus, to within sufficiently high accuracy, $w(t)$ vanishes. By our assumptions $\partial u/\partial x|_{x=0} = 0$, which shows that to within sufficiently high accuracy it is possible to assume that during some time after the passage of the maximum of the boundary of the third phase the temperature of the immersed body and the crystallized crust upon it are practically dependent only on time, remaining spatially constant. It is natural that the duration of the time interval on which this takes place is smaller for a larger flux of heat from the body of the plate imbedded in the melt of the body. At the same time it increases with the thickness of the plate, or with the radius of the sphere.

We also note the a priori evident fact that the duration of the growth of the crust on the body immersed in the melt is considerably less than the duration of its melting. This property, together with the vanishing of w immediately after the passage of the maximum boundary value $x = y(t)$, permits an important reduction in the computing time.

By looking at the graphs we see that in all problems of relevance to engineers the quasi-stationary approximation gives a satisfactory result, which is the more accurate the smaller the heat flux at the immersed body and the larger the dimensions of this body; this justifies the use of the quasi-stationary approximation for approximate calculation.

§3. Dynamics of growth of a bubble of air in water[7)]

Below we examine with precision the quasi-stationary statement of problem $(2.1.4_1)$—$(2.1.4_4)$ of Part 1 for initial conditions (2.1.11). In the quasi-stationary formulation, the problem is reduced to an integral equation (2.1.13) of the first kind with identifying parameter λ and time t. In the exact treatment it is easily shown that the problem is reduced by the method of Part 1, Chapter I to the solution of the system of integral equations

TABLE 10

$p_0 = 1 \cdot 10^6 \text{dyn/cm}$; $\rho_0 = 1.293 \cdot 10^{-3} \text{g/cm}^3$; $\rho_0 = \alpha p_0$; $\alpha = 1.293 \cdot 10^{-9} \text{g/cm}$;

$\sigma = 70 \text{dyn/cm}$; $R_0 = 1 \cdot 10^{-3} \text{cm}$; $\beta = 2\sigma/p_0 R_0 = 10.14$; $b = 2\beta/3 = 0.0933$;

$a = \delta/\alpha = 0.0247$; $c = 0.00346$; $c_s(0) = \delta(p_0 + 2\sigma/R_0) = 3.55 \cdot 10^{-4}$;

$(c_0 - c_s(0))/\delta p_0 = \gamma = \lambda - 1 - \beta$; $\lambda = 0.50;\ 0.75;\ 1.00;\ 1.25;\ 1.50$.

7) See Part 1, Chapter II, §1.

TABLE 11

(Solution of the system (S3.3.1) — y; solution of equation (2.1.13) — y^*)

$\gamma = -0.64$; $\lambda = 0.50$			$\gamma = -20.39$; $\lambda = 0.75$			$\gamma = -0.14$; $\lambda = 1.00$		
t	y	y^*	t	y	y^*	t	y	y^*
0	1	1	0	1	1	0	1	1
1.0	0.972	0.97	1.0	0.982	0.989	1.0	0.996	0.993
2.0	0.950	0.951	2.0	0.969	0.970	2.0	0.991	0.989
5.0	0.892	0.894	5.24	0.931	0.940	5.0	0.978	0.978
8.2	0.830	0.831	9.24	0.885	0.895	10.0	0.958	0.956
12.2	0.794	0.751	12.24	0.850	0.857	20.0	0.918	0.920
17.0	0.643	0.654	16.24	0.802	0.819	30.0	0.876	0.877
22.6	0.495	0.510	18.24	0.777	0.795	40.0	0.831	0.844
27.4	0.313	0.316	23.24	0.711	0.724	50.0	0.782	0.798
29.0	0.210	0.213	27.24	0.653	0.678	60.0	0.726	0.746
29.6	0.151	0.142	31.24	0.589	0.615	70.0	0.662	0.688
30.0	0.077	0.028	39.24	0.428	0.446	80.0	0.585	0.626
			43.24	0.304	0.392	90.0	0.484	0.538
			45.99	0.130	0.207	100.0	0.315	0.420
			46.24	0.082	0.187	110.0	0.009	0.002
			46.50	0.019	0.134			
			47.14	0.000	0.018			

$\gamma = 0.11$; $\lambda = 1.25$			$\gamma = 0.36$; $\lambda = 1.50$		
t	y	y^*	t	y	y^*
0	1	1	0	1	1
5	1.02	1.01	5	1.06	1.06
10	1.03	1.02	10	1.11	1.12
20	1.06	1.11	20	1.20	1.21
30	1.09	1.16	30	1.29	1.30
50	1.15	1.25	50	1.46	1.46
100	1.30	1.47	100	1.83	1.81
200	1.58	1.85	200	2.47	2.39
300	1.85	2.18	300	3.01	2.86
400	2.10	2.39	400	3.53	3.28
500	2.34	2.55	500	4.00	3.65

Remark

t is dimensionless time:

$$t = \frac{D}{R_0^2}\,\tau.$$

$$w(t) = \frac{\gamma \exp\left(-\frac{(y(t)-1)^2}{4t}\right)}{\sqrt{\pi}} + \gamma\sqrt{t}\left(1 - \operatorname{erf}\frac{1-y(t)}{2\sqrt{t}}\right)$$
$$+ 2\beta\sqrt{t}\int_0^t \dot{y}(\tau)E(y(t)-y(\tau), t-\tau)\,d\tau$$
$$- 2\int_0^t w(\tau)\sqrt{\frac{t}{\tau}}\frac{\partial}{\partial x}E(y(t)-y(\tau), t-\tau)\,d\tau; \quad \text{(S3.3.1)}$$
$$y(t) = 1 + \int_0^t \left[\frac{a}{b+y(\tau)}\cdot\frac{w(\tau)}{\sqrt{\tau}} - \frac{c}{y(\tau)}\cdot\frac{1-y(\tau)}{b+y(\tau)}\right]d\tau.$$

The algorithm for the solution of the system (S3.3.1) coincides with the algorithm for the solution of the problem of crystallization of a melt under immersion of a plate, investigated in the previous section. The initial value w is defined as the root of the equation

$$w(0) = \frac{\gamma}{\sqrt{\pi}}\exp(-\mu^2 w^2(0)) + w(0)\operatorname{erf}(\mu w(0)), \quad \text{(S3.3.2)}$$

where $\mu = a/(1+b)$ and a and b are defined in agreement with (2.1.3) of Part 1. The parameters defining the process are shown in Table 10.

TABLE 12

		$\gamma = -0.14$; $\lambda = 1.00$			
		Constant mesh $\Delta t = 1$	Variable mesh		
i	t_i	$y(t_i)$	$y(t_i)$	Δt	On the time interval for t
1	1	0.9956	0.9935	0.01	0— 1
2	2	0.9910	0.9888	0.1	1— 5
3	5	0.9784	0.9760	0.5	5— 10
4	10	0.9584	0.9559	1.0	10— 40
5	20	0.9184	0.9157	5.0	40—100
6	40	0.8314	0.8281		
7	50	0.7819	0.7788		
8	60	0.7264	0.7240		
9	70	0.6625	0.6605		
10	80	0.5853	0.5842		
11	90	0.4838	0.4859		
12	100	0.3151	0.3184		

TABLE 13

Parameter	Component				
	Air $i = -1$	nC_5H_{12} $i = 0$	nC_6H_{14} $i = 1$	nC_7H_{16} $i = 2$	Initial opening
$D_{i1} \cdot 10^5 cm^2/sec$		6.9	5.8	4.9	
$D_{i2} \cdot 10^2 cm^2/sec$		8.4	7.6	7.0	
$d_{i1} \cdot g/cm^2$		0.630	0.660	0.684	0.647
$d_{i2} \cdot 10^3 g/cm^3$	1.293	3.27	3.90	4.55	
M g/mole	29	72	86	100	75.5
$v_{im} cm^3/mole$		114	130	146	
$\sigma^0_{im} \cdot 10^6 mole/cm^3$		32.4	10.2	3.2	
$\lambda_i \cdot 10^3 cal/mole$		6.15	6.82	7.41	
c_{i1} cal/mole · grad		30.0	34.5	40.8	31.6
c_{i2} cal/mole · grad	69		31.4	29.6	
$c_{i0}/c_{00} = N_{i0}$		0.8	0.1	0.1	
$c_{i0} \cdot 10^4 mole/cm^3$		71	7	7	
d g/cm^3					0.647
$\overline{k}_i \cdot 10^4$ cal/cm · sec · grad					2.94
$\overline{k}_2 \cdot 10^4$ cal/cm · sec · grad	5.8				
$b_1^2 \cdot 10^3 cm^2/sec$					1
$b_2^2 \cdot 10^2 cm^2/sec$	2				
$\sigma_{i0} mole/cm^3$		0	0	0	
$\overline{U}_0, \overline{V}_0, U^*$ ° K		293			

$q_1 = q_2 = q^* = 0$; $a_{11} = 1.0$; $a_{21} = 0.919$; $a_{12} = 36.2$; $a_{22} = 34.7$; $a_{02} = 38.0$

The similarity solution[8]

$$u = A_1\left(1 + \operatorname{erf}\frac{x}{2b_1\sqrt{t}}\right); \quad v = A_2\operatorname{erfc}\frac{x}{2b_2\sqrt{t}};$$

(S3.4.1)

$$c_i = A_{i1}\left(1 + \operatorname{erf}\frac{x}{2a_{i1}\sqrt{t}}\right); \quad \sigma_i = A_{i2}\operatorname{erfc}\frac{x}{2a_{i2}\sqrt{t}};$$

$$y = 2\beta\sqrt{t}$$

is determined by the values of the coefficients given in Table 14.

TABLE 14

$A_1 = -0.01651$,	$A_2 = -0.01616$,	$A_{11} = 0.06990$,	$A_{21} = 0.1104$,	
$A_{02} = 0.003018$,	$A_{12} = 0.001250$,	$A_{22} = 0.0004013$,	$\beta = -0.064806$.	

The results of the computation are given in Table 11. The actual computations of y are for the illustration given in Table 12.

The computation of $y(t)$ and $y^*(t)$ for $\lambda > 1$ results in less accuracy; this is due to the necessity of using, together with t, a rapidly increasing sequence of variable mesh lengths, with the aim of decreasing the outlay of machine time. It follows that the reduction in the size of the mesh for $\lambda > 1$ leads to the approach of $y(t)$ and $y^*(t)$ to one another. Granting this, and examining Table 11, we find that the quasi-stationary approximation y^* approximates the limit y with an error of not more than 10% for every case examined and on every interval of time included in Table 11; thus its use is fully justified.

§4. Dynamics of evaporation of a solution of normal sextane and heptane in normal pentane[9]

The dynamics of evaporation of a system nC_5H_{12}: nC_6H_{14}: nC_7H_{16} can be examined approximately (see Part 1, Chapter II, §2). The parameters of the process and initial composition of the system are given in Table 13 [**145**].

[8] The solution is constructed by neglecting the terms c_{i0} comparable to 1.

[9] See Part 1, Chapter II.

In the case under examination the approximation determined by equations (2.2.25), (2.2.26), (2.2.28) and (2.2.29) of Part 1 leads to the relations

$$
\begin{aligned}
u &= b_1\sqrt{\pi}\phi_{10}\operatorname{erfc}\left[\frac{1}{b_1}\left(\frac{x}{2\sqrt{t}} - a_{02}^2\phi_0\right)\right];\\
v &= -b_2\sqrt{\pi}\phi\operatorname{erfc}\left[\frac{1}{b_2}\left(\frac{x}{2\sqrt{t}} - a_{02}^2\phi_0\right)\right];\\
c_i &= a_{i1}\sqrt{\pi}\phi_{1i}\operatorname{erfc}\left[\frac{1}{a_{i1}}\left(\frac{x}{2\sqrt{t}} - a_{02}^2\phi_0\right)\right];\\
\sigma_i &= -a_{i2}\sqrt{\pi}\phi_i\operatorname{erfc}\left[\frac{1}{a_{i2}}\left(\frac{x}{2\sqrt{t}} - a_{02}^2\phi_0\right)\right];\\
y &= 2a_{02}^2\phi_0\sqrt{t}.
\end{aligned}
\qquad \text{(S3.4.2)}
$$

The values appearing here for the coefficients are given in Table 15.

TABLE 15

$\phi_{10} = -0.002238$, $\phi = 0.0005003$, $\phi_{11} = 0.03907$, $\phi_{12} = 0.06721$,
$\phi_0 = -0.4468 \cdot 10^{-4}$, $\phi_1 = -0.1956 \cdot 10^{-4}$, $\phi_2 = 0.6565 \cdot 10^{-5}$,
$a_{02}^2\phi_0 = -0.06470$.

TABLE 16

t, days	y, cm.		t, days	y, cm.	
	(S3.4.2)	(S3.4.1)		(S3.4.2)	(S3.4.1)
2	—53.79	—53.88	12	—131.8	—132.0
4	—76.08	—76.20	14	—142.3	—142.5
6	—93.17	—93.32	16	—152.2	—152.4
8	—107.6	—107.8	18	—161.4	—161.6
10	—120.3	—120.5	20	—170.1	—170.4

In Table 16 we give the values of y calculated from formulas (S3.4.1) and (S3.4.2) for t varying from 0 to 20 days. Thus the relative error in determining y in accordance with (S3.4.2) compared to the determination of y in accordance with (S3.4.1) does not exceed 0.2%; this testifies to the complete suitability of the quasi-stationary approximation for

practical calculation of the dynamic evaporation of a polycomponent mixture in the atmosphere for distribution by diffusion of the vapor concentration.

§5. Zonal noncrucible melting of a cylindrical rod.[10]

We now present the numerical solution of the problem of zonal noncrucible melting of a cylindrical rod, briefly formulated in Part 2, Chapter I, §5. We recall that the rod is moving forward in a field of high frequency induction melting. The problem will be solved by the method of finite differences with the use of the method of Oleĭnik for the construction of a generalized solution for the Stefan problem (see Part 2, Chapter VII, §2); we do not raise the problem of justifying the result.[11]

We will write $\bar{z}$, $\bar{x}$, $\bar{r}$ in place of z, x, r, putting the bar over the dimensioned coordinates. We introduce the dimensionless variables as

$$r = \frac{\bar{r}}{R}; \quad z = \frac{\bar{z}}{R}; \quad x = \frac{\bar{x}}{R}; \quad \tau = \frac{a_1^2}{R^2} t;$$

(S3.5.1$_1$)
$$\mu = \frac{vR}{a_1^2}; \quad b = \frac{\lambda \rho a_1^2}{k_1 T_п}; \quad \gamma = \frac{\sigma R T_п^3}{k_1};$$

$$k(\xi) = \begin{cases} 1; & \xi < 0 \\ \dfrac{k_1}{k_2}; & \xi > 0; \end{cases} \qquad \psi(x) = \frac{R}{k_1 T_п} \Phi(Rx);$$

(S3.5.1$_2$)
$$u = \frac{\bar{u} - \bar{u}_п}{\bar{u}_п}; \quad \bar{u} = \begin{cases} k_1 T & \text{for } T < T_п; \\ k_1 T_п + k_2 (T - T_п) & \text{for } T > T_п; \end{cases}$$

(S3.5.1$_3$)
$$a(u) = \begin{cases} u & \text{for } u < 0, \\ \dfrac{k_1}{k_2} u + b & \text{for } u > 0. \end{cases}$$

In terms of the variables (S3.5.1$_1$)—(S3.5.1$_3$) and with the aid of the method described in Part 2, Chapter VII, §2, problem (1.5.3$_i$) of Part 1 results in the problem

(S3.5.2$_1$)
$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial a(u)}{\partial \tau};$$

$$0 < r < 1; \quad -\infty < z < \infty; \quad u \neq 0; \quad \tau > 0;$$

[10] See Part 1, Chapter I, §5; Part 2, Chapter VII, §2 and Chapter VIII, §2.

[11] Such a justification is necessary, since the domain of definition of the solution is unbounded and the boundary condition at the surface of the rod is nonlinear.

$$(S3.5.2_2) \qquad -\frac{\partial u}{\partial r}\bigg|_{r=1} = \gamma[k(u)u+1]^4 - \psi(x); \quad x = z + \mu\tau;$$

$$(S3.5.2_3) \qquad \lim_{|z|\to\infty} \frac{\partial u}{\partial z} = 0; \quad 0 \leqq r \leqq 1.$$

The solution u should be bounded for $r = 0$. The condition of boundedness can be replaced by the condition

$$(S3.5.2_4) \qquad \frac{\partial u}{\partial r}\bigg|_{r=0} = 0; \quad -\infty < z < \infty; \quad \tau \geqq 0.$$

The initial condition is not written, since our interest is in the asymptotic behavior of the solution for $\tau \to \infty$.

We pass to the stationary system of coordinates x, r, τ, fixed to the inductor, and simultaneously we note that the problem, posed for an unbounded domain, can be formulated for a bounded domain with the additional boundary condition

$$(S3.5.2_5) \qquad \frac{\partial u}{\partial x} = 0 \text{ for } |x| = B > 0; \quad 0 \leqq r \leqq 1; \quad \tau > 0.$$

It is obvious that such a reduction of the domain $-\infty < x < \infty$ is permissible if $B > 0$ is sufficiently large. Thus we arrive at the problem

$$(S3.5.3_1) \qquad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial}{\partial \tau} + \mu\frac{\partial}{\partial x}\right) a(u); \quad -B < x < B;$$
$$0 < r < 1; \quad u \neq 0; \quad \tau > 0;$$

$$(S3.5.3_2) \qquad -\frac{\partial u}{\partial r}\bigg|_{r=1} = \gamma(k(u)u+1)^4 - \psi(x); \quad -B < x < B; \quad \tau > 0;$$

$$(S3.5.3_3) \qquad \frac{\partial u}{\partial r}\bigg|_{r=0} = 0; \quad \frac{\partial u}{\partial x}\bigg|_{|x|=B} = 0 \text{ for } \tau \geqq 0.$$

$\psi(x)$ in $(S3.5.3_2)$ is assumed given as

$$(S3.5.4) \qquad \psi(x) = \frac{p\left[\operatorname{arctg}\dfrac{\bar{x}+a/2}{d} - \operatorname{arctg}\dfrac{\bar{x}-a/2}{d}\right]\dfrac{R}{k_1 T_{\text{п}}}}{2\pi R\displaystyle\int_{-\infty}^{\infty}\left(\operatorname{arctg}\frac{\bar{x}+a/2}{d} - \operatorname{arctg}\frac{\bar{x}-a/2}{d}\right)^2 d\bar{x}}.$$

Here $p = \text{const} > 0$ is the power radiated by the inductor, a is the

diameter of a cross-section of the inductor and d is the magnitude of the tolerance between the inductor and the rod.[12] Following the method of Oleĭnik (see Part 2, Chapter VII), we smooth the jump function $a(u)$, i.e. we replace it by the polygon

$$a^*(u) = \begin{cases} a(u); & |u| \in (-L, \infty); \\ a(u) + \beta; & u \in (-L, L) \end{cases} \tag{S3.5.5}$$

by using a smoothing of it in a neighborhood of the vertex (see Part 2, Chapter VIII, §2C). We set

$$c(u) = \frac{da^*}{du}; \quad u \neq \pm L, \tag{S3.5.6$_1$}$$

so that outside of a small neighborhood of the point $u = \pm L$

$$c(u) = \begin{cases} 1; & u < -L; \\ \dfrac{b}{2L} + \dfrac{1}{2}\left(1 + \dfrac{k_1}{k_2}\right); & -L < u < L; \\ \dfrac{k_1}{k_2}; & u > L. \end{cases} \tag{S3.5.6$_2$}$$

Thus we arrive at the problem

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2}\right)u = c(u)\left(\frac{\partial}{\partial \tau} + \mu\frac{\partial}{\partial x}\right)u$$

$$\text{for } 0 < r < 1; \quad -B < x < B; \quad \tau > 0;$$

$$-\frac{\partial u}{2r}\bigg|_{r=1} = [\gamma(k(u)u + 1)^4 - \psi(x)]_{r=1}; \tag{S3.5.7}$$

$$-B < x < B; \quad \tau > 0;$$

$$\frac{\partial u}{2r}\bigg|_{r=0} = 0; \quad \frac{\partial u}{\partial x}\bigg|_{|x|=B} = 0; \quad \tau > 0.$$

Since it is required to find only the asymptotic behavior of the solution for $\tau \to \infty$, the stationary problem obtained from problem (S3.5.7) can be solved at once, if in it we set $\partial u/\partial \tau \equiv 0$. This problem can be solved by the method of finite differences with the use of an iteration scheme in pure form or an iteration scheme (the application of which

[12] (S3.5.4) gives the power of the energy being radiated over the conducting surface [5], which for sufficiently large inductor radius satisfactorily approximates the energy distribution at the surface of the rod.

is necessary by virtue of the nonlinearity of the problem) in combination with the "matrix sweep" method. It is known that a pure iteration scheme for the solution of an elliptic equation converges extremely slowly. The "matrix sweep" method imposes rigid limitations on the maximum number of mesh points which can be used without conversion to exterior memory on the ÈVCM. For the many-invariant computation, when it is required that we obtain as large a reduction in the machine time as possible, a computing algorithm is desired which can be used without passage to the external memory of the ÈVCM. In our case this goal is attained by the solution of the finite-difference analog of the system (S3.6.7) with the use of one of the versions of the method of splitting of a two-dimensional difference parabolic operator into one-dimensional operators.

Two schemes have been tested: the scheme of Peaceman and Rachford [111] and the scheme recommended by A. A. Samarskiĭ [152], [153]. The first of these schemes, especially intended for the solution of the linear stationary problem, seems suitable when the mesh length in time is chosen not too large. In correspondence to the theory of stationary conditions, this asymptotically defined solution proves to be practically independent of the choice of the size of the time mesh. A limitation on the value of the time step is still needed for the sake of stability.

An additional check on the applicability of the method is the following. In the case $\mu = 0$ the stationary temperature distribution is not dependent on the choice of the value of L entering into the expression (S3.5.6$_2$) for the heat content. In the method of Peaceman-Rachford this condition is satisfied (see Table 18, below).

The stationary conditions attained by the use of Samarskiĭ's scheme are found to be dependent on the value of the time step, and thus for its application one must necessarily use a much smaller mesh than is acceptable for the Peaceman-Rachford scheme, in spite of the greater stability of Samarskiĭ's scheme as compared to their scheme. At the same time, the stationary conditions attained by using Samarskiĭ's scheme turn out to be essentially dependent on the value of L for $\mu = 0$ (see Table 18). Thus the scheme of Samarskiĭ is unacceptable for the solution of the problem under consideration. Therefore the calculation is conducted by the Peaceman-Rachford method.

We pass to the details of the algorithm and the resulting calculations. We introduce a difference mesh with the space nodes at the points (x_i, r_j) and the time nodes at the points τ_k, with $i = 0, \cdots, M$; $x_0 = -B$; $x_M = B$; $j = 0, \cdots, N$; $r_0 = 0$, $r_N = 1$; $k = 0, 1, \cdots$; $\tau_0 = 0$. We set

$$u^k_{i,j} = u(x_i, r_j, t_k); \quad c^k_{i,j} = c(u^k_{i,j});$$
$$h_i = x_i - x_{i-1}; \quad g = r_j - r_{j-1} = \text{const}; \quad l_k = \tau_k - \tau_{k-1}. \tag{S3.5.8}$$

In addition to the time steps with integral indices, we introduce time steps with semi-integral indices. In agreement with the Peaceman-Rachford scheme the first of the equations (S3.5.7) is replaced by the equations

$$\frac{c^k_{i,j}}{l_k}(u^{k+1/2}_{i,j} - u^k_{i,j}) + \frac{\mu c^k_{i,j}}{h_i + h_{i+1}}(u^{k+1/2}_{i+1,j} - u^{k+1/2}_{i-1,j})$$
$$= \frac{1}{g^2}(u^k_{i,j-1} - 2u^k_{i,j} + u^k_{i,j+1}) + \frac{1}{2r_j g}(u^k_{i,j+1} - u^k_{i,j-1}) \tag{S3.5.9$_1$}$$
$$+ \frac{2u^{k+1/2}_{i+1,j}}{h_{i+1}(h_i + h_{i+1})} + \frac{2u^{k+1/2}_{i-1,j}}{h_i(h_i + h_{i+1})} - \frac{2u^{k+1/2}_{i,j}}{h_i h_{i+1}};$$

$$\frac{c^{k+1/2}_{i,j}}{l_k}(u^{k+1}_{i,j} - u^{k+1/2}_{i,j}) + \frac{\mu c^{k+1/2}_{i,j}}{h_i + h_{i+1}}(u^{k+1/2}_{i+1,j} - u^{k+1/2}_{i-1,j})$$
$$= \frac{1}{g^2}(u^{k+1}_{i,j-1} - 2u^{k+1}_{i,j} + u^{k+1}_{i,j+1}) + \frac{1}{2r_j g}(u^{k+1}_{i,j+1} - u^{k+1}_{i,j-1}) \tag{S3.5.9$_2$}$$
$$+ \frac{2u^{k+1/2}_{i+1,j}}{h_{i+1}(h_i + h_{i+1})} + \frac{2u^{k+1/2}_{i-1,j}}{h_i(h_i + h_{i+1})} - \frac{2u^{k+1/2}_{i,j}}{h_i h_{i+1}}.$$

The boundary condition on the plane $r = 1$ is approximated by the condition

$$\frac{1}{g}(u^{k+1}_{i,N} - u^{k+1}_{i,N-1}) = \psi_i - \gamma(k(u^k_{i,N})u^k_{i,N} + 1)^3(k(u^k_{i,N})u^{k+1}_{i,N} + 1). \tag{S3.5.9$_3$}$$

The condition $\partial u/\partial r = 0$ for $r = 0$ is replaced by the condition

$$2\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial \tau} + \mu\frac{\partial u}{\partial x} \text{ for } r = 0, \tag{S3.5.9$_4^*$}$$

implied by the equation (S3.5.7), since, by virtue of the fact that $\partial u/\partial r = 0$ for $r = 0$, we have

$$\lim_{r\to 0}\frac{1}{r}\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial r^2}\bigg|_{r=0}.$$

The condition (S3.5.9$_4^*$) is approximated by a difference expression analogous to the expression (S3.5.9$_2$) but with $u^{k+1}_{i,-1}$ replaced by $u^{k+1}_{i,1}$. The analogous condition $\partial u/\partial x|_{|x|=B} = 0$ is replaced by the condition

(S3.5.9$_1$) for $i = 0$ and $i = N$, but with the replacement in it of $u_{-1,j}^{k+1/2}$ and $u_{+1,j}^{k+1/2}$ by $u_{1,j}^{k+1/2}$ and $u_{-1,j}^{k+1/2}$ respectively.

The system (S3.6.9$_i$) is solved by sweeping in the direction of the x-axis in passage from a time plane with an integer index to one with a fractional index, and for passage from a fractional plane to an integer plane it is solved by sweeping in the direction of the r-axis.

On every interval of the variable x we have used 80 steps. The value B is chosen such that the shape of the rod does not lead to essentially distorted isotherms in the domain of influence of the inductor. Finally B is taken equal to 13.5. The interval $-13.5 < x < 13.5$ is subdivided by points x_i such that

$$h_i = \begin{cases} 1 & \text{for } i = 1, 2, \cdots, 6; \\ 1/2 & \text{for } i = 7, 8, \cdots, 16; \\ 1/6 & \text{for } i = 17, 18, \cdots, 22; \\ 1/12 & \text{for } i = 23, 24, \cdots, 58; \\ 1/6 & \text{for } i = 59, 60, \cdots, 64; \\ 1/2 & \text{for } i = 65, 66, \cdots, 74; \\ 1 & \text{for } i = 75, 76, \cdots, 80. \end{cases}$$

The step size in r is constant, and taken as $g = 1/12$. The value L, which determines the character of the approximation $a(u)$, is subject to change. L was chosen as $L = 0.0125$, which guarantees that each path is located in an interval of 2 or 3 mesh nodes. In Figure 10 we show the form of the isotherm obtained in one of the typical calculations of variants for $L = 0.025$ and $L = 0.0125$. For comparison we show in

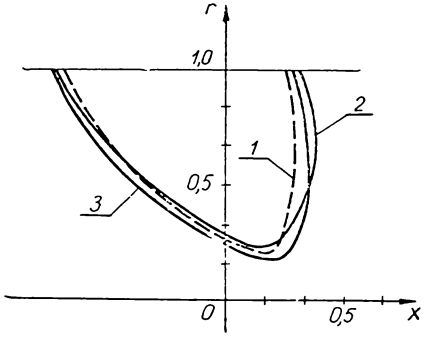

FIGURE 10. Melting isotherms.
1—$L = 0.025$; 2—$L = 0.0125$; 3—Without smoothing $a(u)$, and with an explicit difference scheme.

the same figure the isotherm for the melting found by means of the solution to the problem with the use of an explicit difference scheme. All three curves are sufficiently similar for us to regard the approximation L as satisfactory for carrying out an engineering calculation.

TABLE 17

Parameters	Remarks
$\alpha = 0.4$ cm; $d = 0.5$ cm	See (S3.5.4). The parameter p changes from case to case.
$c = 0.96$ volt · sec/g · grad $k_1 = 0.3$ volt/cm · grad $\rho = 2.3$ g/cm^3 $\lambda = 1800$ volt · sec/g	See definition (S3.5.1). The radius R of the rod, its velocity v and the ratio of the coefficients of conductivity of the solid and liquid phases k_1/k_2 change from case to case.
$\sigma = 3.6 \cdot 10^{-12}$ volt/cm^2 · grad	σ includes a correction for dullness

TABLE 18

x	$L = 0$			$L = 0.05$			Remarks
	U_A	U_B	U_C	U_A	U_B	U_C	
0	0.031	0.025	0.028	0.006	0.006	0.027	The calculation
1/12	0.028	0.022	0.026	0.005	0.004	0.026	is performed
2/12	0.022	0.016	0.022	0.000	0.000	0.022	for time step
3/12	0.012	0.007	0.015	—0.007	—0.008	0.014	$l_k = 0.025$;
4/12	—0.001	—0.004	0.005	—0.017	—0.018	0.005	
5/12	—0.015	—0.018	—0.007	—0.030	—0.030	—0.007	$k_1/k_2 = 1$;
6/12	—0.031	—0.033	—0.020	—0.044	—0.044	—0.020	
7/12	—0.048	—0.050	—0.035	—0.056	—0.057	—0.035	$R = 1.5$ cm;
8/12	—0.066	—0.067	—0.051	—0.073	—0.074	—0.051	
9/12	—0.084	—0.084	—0.068	—0.090	—0.091	—0.068	$p = 1700$ volt

All the calculations have been conducted for silicon. The numerical parameter values entering into the calculations are given in Table 17.

In Table 18 we show the results of comparison of the effect of inclusion in the calculation of the smoothed unit function $a(u)$ in the schemes of Samarskiĭ and Peaceman-Rachford. In the tabulation U_A, U_B and U_C denote the values of the stationary temperature on the axis of the rod, evaluated by Samarskiĭ's scheme (U_A and U_B)

and by the Peaceman-Rachford scheme (U_C). Here U_A is obtained as the limit for $k \to \infty$ of the values $U_{i,0}$ for k an integer, and U_B, the limit of the half-sum $\frac{1}{2}(u_{ip}^k + u_{i0}^{k+1/2})$.[13] The column $L = 0$ corresponds to the truncation for smoothed $a(u)$. The columns $L \neq 0$ are truncated.[14]

As we see from the table, the truncation for the smoothed function is not found to have an effect on the value of U_C within three digits. Conversely, the values U_A and U_B change very much under a change of $L = 0$ to $L = 0.05$. In particular, the change in L displaces the position of the isotherm of the melt, which really exhibits the inadmissibility of using Samarskii's scheme for the solution of the problem of zonal noncrucible melting.

In Table 19 we have tabulated the values of the temperature $u(x, r)$ computed by the Peaceman-Rachford method with the mesh nodes belonging to a domain occupied by the melt. The calculation is conducted for $R = 1.5\,\text{cm}$, $p = 1700$ volts, $v = 5\,\text{mm/min}$ and $k_1/k_2 = 1$.

In carrying out the computation it is assumed that the steady state is reached if the maximum difference in the values of the temperature for two successive integer meshes is not greater than 10^{-3}. To determine the form of the isotherm of melting ($u = 0$), a linear interpolation between two adjacent lattice points is constructed. This introduces an additional error in determining the unknown isotherm, which differs from case to case. In those cases when the minimum distance between the isotherms of the melt and of the crystallization is sufficiently large, the exact construction of the isotherm turns out to be satisfactory. In those cases when the isotherms of the liquid and solid come together, i.e. in the case similar to the appearance or vanishing of the continuous melt of the rod, the accuracy sharply decreases. This indicates the necessity of concentrating the mesh, particularly in the domain of the phase transition.

The character of dependence of the form and position of the isotherms of melting and of crystallization on the speed of motion of the rod is shown in Figures 11 and 12. Qualitatively the picture obtained is the

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[13] For the Dirichlet problem of Laplace's equation in a rectangular domain, the value U_B automatically gives a more accurate result than does U_A.

[14] In carrying out the calculation of the smoothed $a(u)$ in a neighborhood of a vertex of the polygon, (S3.5.5) is not used. Instead, we take $c(u) = c(u - 0)$. In this regard, the probability of assuming the values $u = \pm L$ at a node of a small mesh for such a smoothing is practically zero; hence this has no effect on the resulting computations.

TABLE 19

x	r												
	0	1/12	1/6	1/4	1/3	5/12	1/2	7/12	2/3	3/4	5/6	11/12	1
—3/4	—0.105	—0.105	—0.104	—0.104	—0.103	—0.102	—0.102	—0.102	—0.101	—0.101	—0.102	—0.104	—0.106
—2/3	—0.090	—0.090	—0.089	—0.088	—0.087	—0.085	—0.085	—0.085	—0.084	—0.083	—0.083	—0.083	—0.085
—7/12	—0.076	—0.076	—0.075	—0.074	—0.073	—0.071	—0.070	—0.067	—0.065	—0.063	—0.062	—0.062	—0.063
—1/2	—0.063	—0.063	—0.062	—0.061	—0.059	—0.057	—0.054	—0.051	—0.047	—0.044	—0.041	—0.039	—0.038
—5/12	—0.050	—0.050	—0.048	—0.047	—0.045	—0.042	—0.038	—0.033	—0.028	—0.021	—0.015	—0.008	—0.005
—1/3	—0.039	—0.038	—0.037	—0.035	—0.033	—0.029	—0.025	—0.019	—0.013	—0.004	0.006	0.017	0.025
—1/4	—0.028	—0.028	—0.027	—0.025	—0.022	—0.018	—0.013	—0.005	0.005	0.019	0.033	0.049	0.065
—1/6	—0.020	—0.020	—0.018	—0.016	—0.013	—0.008	—0.001	0.009	0.023	0.039	0.058	0.080	0.107
—1/12	—0.013	—0.013	—0.011	—0.009	—0.005	—0.002	0.011	0.023	0.039	0.055	0.077	0.105	0.142
0	—0.008	—0.007	—0.005	—0.002	0.003	0.010	0.020	0.032	0.047	0.065	0.088	0.118	0.159
1/12	—0.003	—0.002	0.000	0.004	0.009	0.017	0.026	0.037	0.051	0.068	0.090	0.117	0.153
1/6	0.002	0.002	0.004	0.008	0.013	0.020	0.028	0.038	0.050	0.065	0.082	0.104	0.130
1/4	0.004	0.004	0.007	0.010	0.014	0.020	0.027	0.035	0.045	0.056	0.069	0.084	0.099
1/3	0.005	0.005	0.007	0.010	0.013	0.017	0.023	0.029	0.036	0.045	0.053	0.062	0.069
5/12	0.003	0.004	0.005	0.007	0.010	0.013	0.018	0.021	0.026	0.031	0.037	0.041	0.043
1/2	—0.001	—0.001	0.001	0.002	0.004	0.007	0.009	0.012	0.015	0.018	0.021	0.022	0.021
7/12	—0.008	—0.008	—0.007	—0.006	—0.005	—0.003	—0.001	0.001	0.003	0.004	0.005	0.004	0.002
2/3	—0.021	—0.020	—0.020	—0.019	—0.018	—0.017	—0.016	—0.015	—0.014	—0.014	—0.015	—0.016	—0.020
3/4	—0.033	—0.033	—0.033	—0.032	—0.032	—0.032	—0.031	—0.031	—0.031	—0.032	—0.033	—0.036	—0.040
5/6	—0.047	—0.047	—0.047	—0.046	—0.046	—0.046	—0.046	—0.047	—0.047	—0.049	—0.051	—0.054	—0.058

same as that found experimentally. The results of the calculations cannot be claimed as quantitatively agreeing with the experimental results, since the effective values of the coefficients of heat conduction in the liquid phase are unknown. In addition, the calculation cannot claim to be highly accurate, since only a rough mesh is used. The use of this method can serve as a quick examination of the possibility of a computational algorithm for solving the problem, but not as a convincing quantitative characterization of the process.

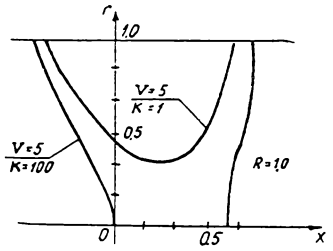

FIGURE 11. Melting isotherms: R in cm; V in mm/min; $k = k_1/k_2$.

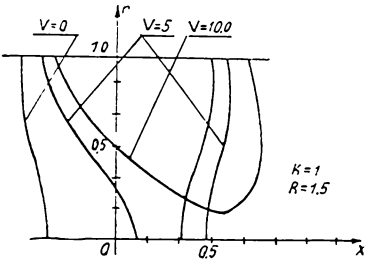

FIGURE 12. Melting isotherms: R in cm; V in mm/min; $k = k_1/k_2$.

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